MATH2011 Tutorial Notes #01
(Introduction to Multivariable Calculus)

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Problem 1. *The outer circle in the Figure 1 has radius* 1 *and the centers of the interior circular arcs lie on the outer circle. Find the area of the shaded region.*

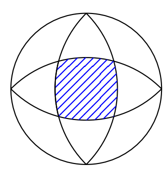

Figure 1: Problem 1.

Solution. As shown in the Figure 2, only $\frac{1}{8}$ of the area needs to be calculated due to symmetry.

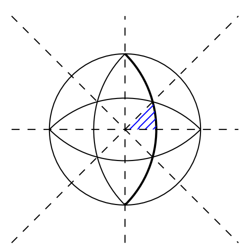

Figure 2: Solution to Problem 1.

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The circle with center $(-1, 0)$ and radius $\sqrt{2}$ (where the thick arc lies) has equation

$$(x+1)^2 + y^2 = 2, \tag{1}$$

converting to polar coordinate,

$$(r\cos\theta + 1)^2 + (r\sin\theta)^2 = 2. \tag{2}$$

The equation above can be simplified to be a quadratic equation with respect to r,

$$r^2 + 2r\cos\theta = 1. \tag{3}$$

Using the quadratic formula to solve for r gives

$$r = \frac{-2\cos\theta \pm \sqrt{4\cos^2\theta + 4}}{2}, \tag{4}$$

and since we presume that $r > 0$,

$$r = -\cos\theta + \sqrt{\cos^2\theta + 1}. \tag{5}$$

Then, denoting the area of the shaded region in Figure 1 to be A, we have

$$\frac{1}{8}A = \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta. \tag{6}$$

Plug Equation (5) into Equation (6) and calculate the integral, yielding

$$A = \frac{2\pi}{3} + 2 - 2\sqrt{3}. \tag{7}$$

□

Problem 2. *Show that any tangent line to a hyperbola touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.*

Solution. Without loss of generality, assume the hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{8}$$

Take derivatives on both sides with respective to x, yielding

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0, \tag{9}$$

therefore

$$y' = \frac{b^2 x}{a^2 y}. \tag{10}$$

The tangent line at the point (c, d) on the hyperbola has equation (assuming $y \neq 0$ here, and if $y = 0$, the conclusion clearly holds since all three points coincide):

$$y - d = \frac{b^2 c}{a^2 d}(x - c). \tag{11}$$

Plugging into equations of asymptotes yields that the tangent line intersects the asymptote $y = \frac{b}{a}x$ at $\left(\frac{ad+bc}{b}, \frac{ad+bc}{a}\right)$, and it intersections the asymptote $y = -\frac{b}{a}x$ at $\left(\frac{bc-ad}{b}, \frac{ad-bc}{a}\right)$.

Then the midpoint of these intersection points is exactly (c, d), the point of tangency.

$\square$