Math 2352—Spring 2016

Week 02 Worksheet: Linear first-order equations and applications

To receive credit, hand in as many solved practice problems as time permits. Complete all unfinished problems at home. Solution of this worksheet will be made available on the website.

1. **(Demonstration)** Model the following first-order linear ode for $y = y(x)$:

   $$xy' + y = e^x, \qquad y(1) = 0.$$

2. **(Practice)** Solve linear odes for $y = y(x)$.
   (a) $x^2y' = 1 - 2xy, \quad y(1) = 2$
   (b) $x^4y' + 4x^3y = e^{-x}, \quad y(1) = -1/e$
   (c) $y' + 2xy = x, \quad y(0) = 1/2$
   (d) $(1 + x^2)y' + 2xy = 2x, \quad y(0) = 0$
   (e) $y' + \lambda y = a + be^{-\lambda x}, \quad y(0) = 0 \quad (\lambda > 0)$

3. **(Practice)** Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.
   (a) Find the T required for the original sum to double in value as a function of r.
   (b) Determine T if $r = 7\%$.
   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

4. **(Practice)** A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.
   (a) Determine the maximum amount that this buyer can afford to borrow.
   (b) Determine the total interest paid during the term of the mortgage.

5. **(Practice)** Find the escape velocity for a body projected upward with an initial velocity v_0 from a point $x_0 = \xi R$ above the center of the earth, where R is the radius of the earth and ξ is a constant greater than unity. Neglect air resistance. Find the initial altitude from which the body must be launched in order to reduce the escape velocity to 85% of its value at the earth's surface.

6. **(Practice)** A tank initially contains an amount S (liters) of pure water. A mixture containing a concentration γ (grams/liter) of salt enters the tank at a rate r (liters/minute), and the well-stirred mixture leaves the tank at the same rate.
   (a) Determine a differential equation for the amount of salt $M(t)$ (grams) in the tank at any time t by writing an equation for $M(t + \Delta t)$.
   (b) Solve this differential equation using an integrating factor.
   (c) Find the limiting amount of salt in the tank as $t \to \infty$, and show that this corresponds to the solution obtained by setting $dM/dt = 0$.

7. **(Practice)** A spherical raindrop evaporates at a rate proportional to its surface area. By finding and solving a differential equation for the radius of the drop, show that the radius decreases linearly with time. Use the following facts to help you solve this problem:

   (i) The volume of a raindrop is $\frac{4}{3}\pi r^3$.

   (ii) The surface area of a raindrop is $4\pi r^2$.

   (iii) The time-derivative of the volume is proportional to the surface area.