Solution to Week 11 Worksheet: Fourier series

1. **(Demonstration)**
2. **(Practice)** Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0; \\ 1, & 0 < x \le \pi. \end{cases}$$

(Hint: Use the result from Q1 to find the Fourier series for $g(x) = f(x) - 1/2$.)

From Q1, we have the result that

$$g(x) = \begin{cases} -1, & -\pi \le x \le 0; \\ 1, & 0 < x \le \pi. \end{cases}$$

can be represented as

$$g(x) = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin(2k-1)x.$$

And we have

$$f(x) = \frac{1}{2}[g(x) + 1].$$

Therefore, we have

$$f(x) = \frac{1}{2}[\sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin(2k-1)x + 1]$$

Or simply

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)} \sin(2k-1)x.$$

3. **(Practice)** Find the Fourier series of the sawtooth function

$$f(x) = x, \quad -\pi < x \le \pi.$$

Since $f(x)$ is odd, it can be represented by the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The coefficient b_n are

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, \mathrm{d}x \\ &= -\frac{2}{n} \cos n\pi \end{aligned}$$

Since

$$\cos n\pi = \begin{cases} -1, & \text{if n odd;} \\ 1, & \text{if n even,} \end{cases}$$

We have

$$b_n = \begin{cases} \frac{2}{n}, & \text{if n odd;} \\ -\frac{2}{n}, & \text{if n even.} \end{cases}$$

Therefore, the Fourier series for $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx.$$

4. **(Practice)** Find the Fourier series of the function

$$f(x) = x^2, \quad -\pi < x \leq \pi.$$

Because $f(x)$ is even, it can be represented by the Fourier cosin series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx.$$

The coefficient a_0 is

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \mathrm{d}x \\ &= \frac{2}{3} \pi^2 \end{aligned}$$

The coefficient for $n > 0$ are

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \mathrm{d}x \\ &= \frac{4}{n^2} \cos n\pi \end{aligned}$$

Therefore the Fourier series for $f(x)$ is given by

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$