Solution to Worksheet 10: Fixed points and linear stability

1. **(Demonstration)**
2. **(Demonstration)**
3. **(Practice)** Find all the fixed points of the following odes and classify their stability:

(a) $\dot{x} = 4x^2 - 16$

At fixed points, $x' = 0$, which implies $x_\star = 2$ or $x_\star = -2$.
Stability of these equilibrium points may be determined by considering the derivative of $f(x) = 4x^2 - 16$. We have $f'(x) = 8x$. Therefore, $f'(2) = 16 > 0$ so that $x_\star = 2$ is an unstable fixed point, and $f'(-2) = -16 < 0$ so that $x_\star = -2$ is a stable fixed point.

(b) $\dot{x} = \sin x$

$x' = 0$ implies $x_\star = n\pi$, where $n \in \mathbb{N}$.
If n is even, or $n = 2k$, then $f'(2k\pi) = cos(2k\pi) = 1 > 0$, so that $x_\star = 2k\pi$ are unstable points.
If n is odd, or $n = 2k+1$, then $f'((2k+1)\pi) = cos((2k+1)\pi) = -1 < 0$, so that $x_\star = (2k+1)\pi$ are stable points.

(c) $\dot{x} = x^2(6 - x)$

$x' = 0$ implies $x_\star = 0$ or $x_\star = 6$.
$f(x) = x^2(6-x)$, then $f'(x) = 12x - 3x^2$. Therefore $f'(6) = 72 - 108 < 0$, so that $x_\star = 6$ is stable point. $f'(0) = 0$ so that $x_\star = 0$ is marginally stable.

Remark: If the fixed point is marginally stable, the next higher-order term in the Taylor series expansion must be considered. In the last exercise, we found $f'(0) = 0$ and $f''(0) = 12 \neq 0$. Therefore Taylor series expanding about $\epsilon = 0$, we have

$$\begin{aligned} \epsilon' &= f(x_\star + \epsilon) \\ &= f'(x_\star) + \epsilon f'(x_\star) + \frac{\epsilon^2}{2} f''(x_\star) + \dots \\ &= 6\epsilon^2 + \dots \end{aligned}$$

The omitted terms in the Taylor series expansion are proportional to ϵ^3 and can be made negligble over a short time interval with respect to the kept term, by taking $\epsilon(0)$ sufficiently small. Therefore, the differential equation over time interval would be approximated by $\epsilon' = 6\epsilon^2$, which has by now the solution

$$\epsilon(t) = \frac{\epsilon(0)}{1 - 6t\epsilon(0)}.$$

If $\epsilon(0) < 0$, then the perturbation of the fixed point decays. However if $\epsilon(0) > 0$, the perturbation will increasing. Therefore this fixed point is unstable.

4. **(Practice)** Find all the fixed points of $\dot{x} = x(3 - 2x - y)$, $\dot{y} = y(2 - x - y)$ and classify their stability.

The fixed points are determined by solving

$$f(x, y) = x(3 - 2x - y) = 0, \qquad g(x, y) = y(2 - x - y) = 0.$$

There are four fixed points $(x_\star, y_\star) : (0, 0), (0, 2), (\frac{3}{2}, 0)$ and $(1, 1)$. The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 4x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}.$$

With $\boldsymbol{J}_\star$ the Jacobian matrix evaluated at the fixed point, we have

$$(x_\star, y_\star) = (0, 0) : \qquad \boldsymbol{J}_\star = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{J}_\star$ are $\lambda = 3, 2$ so that the fixed point $(0, 0)$ is an unstable node. Next,

$$(x_\star, y_\star) = (0, 2) : \qquad \boldsymbol{J}_\star = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{J}_\star$ are $\lambda = 1, -2$ so that the fixed point $(0, 2)$ is an unstable saddle node. Next,

$$(x_\star, y_\star) = (\frac{3}{2}, 0) : \qquad \boldsymbol{J}_\star = \begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{J}_\star$ are $\lambda = -3, \frac{1}{2}$ so that the fixed point $(\frac{3}{2}, 0)$ is an unstable node. Finally,

$$(x_\star, y_\star) = (1, 1) : \qquad \boldsymbol{J}_\star = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}.$$

The eigenvalues of $\boldsymbol{J}_\star$ satisfies $\lambda_1 + \lambda_2 = tra(\boldsymbol{J}_\star) = -3, \ \lambda_1\lambda_2 = det(\boldsymbol{J}_\star) = 1$ thus $\lambda_1 < 0, \lambda_2 < 0$ so that the fixed point $(1, 1)$ is a stable node.