Solutions to "Week 09 Worksheet"*

Normal Modes

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1. **(Demonstration)** Review how to derive the governing equations, determine the normal modes, and write the general solution of the following spring-mass problem discussed in lecture:

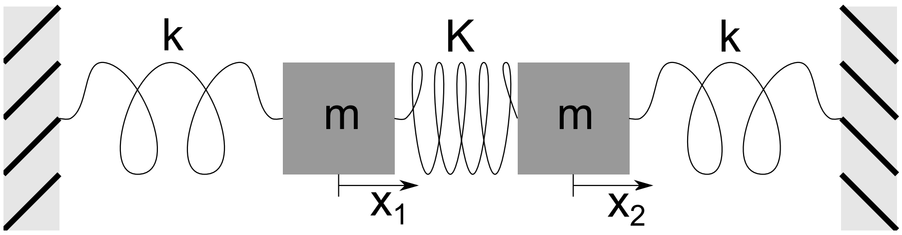

Figure 1. Two masses, three springs

Solution:

To derive the governing equation for the coupled oscillators, we first notice that the instantaneous state of the system is conveniently specified by the displacements (restricted to 1D) of the masses, $x_1(t)$ and $x_2(t)$, respectively. The extensions (compared to equilibrium) of the left, middle and right spings are x_1, $x_2 - x_1$, and $-x_2$, respectively, assuming that $x_1 = x_2 = 0$ correspondes to the equilibrium configuration.

The equations of motion of the two masses are thus

$$\begin{cases} m\,\ddot{x}_1 = -k\,x_1 + K(x_2 - x_1), \\ m\,\ddot{x}_2 = -K(x_2 - x_1) + k(-x_2). \end{cases}$$

Here we are assuming that the springs are of linear elasticity, and ignoring springs' inertia. Therefore, a mass attached to the left end of a spring of extension x and spring constant k experiences a horizontal force $k\,x$, whereas a mass attached to the right end of the same spring experiences an equal and opposite force $-k\,x$.

The equations can be rewritten in the form

$$\begin{cases} \ddot{x}_1 = -\dfrac{k+K}{m} x_1 + \dfrac{K}{m} x_2, \\ \ddot{x}_2 = \dfrac{K}{m} x_1 - \dfrac{k+K}{m} x_2. \end{cases}$$

*. This document has been written using the GNU $\mathrm{T_EX_{MACS}}$ text editor (see `www.texmacs.org`).

Besides solving the system via general tools discussed before, we may make use of symmetry of this problem to decouple the system

$$\begin{cases} \ddot{x}_1 + \ddot{x}_2 &= -\dfrac{k}{m}(x_1 + x_2), \\ \ddot{x}_1 - \ddot{x}_2 &= -\dfrac{k+2K}{m}(x_1 - x_2). \end{cases}$$

The insight provided in doing so is that, for any arbitrary motion of the system, the quantity $x_1 + x_2$ oscillates with frequency $\omega_1 = \sqrt{\frac{k}{m}}$, and the quantity $x_1 - x_2$ oscillates with frequency $\omega_2 = \sqrt{\frac{k+2K}{m}}$. In other words, we have found the *normal coordinates* of the system. The motion of the system can be decomposed into two independent parts.

After obtaining the equations for normal coordinates, the *normal modes* can be calculated by setting one of the normal coordinates to be zero:

When $x_1 - x_2 = 0$,

$$\begin{cases} \ddot{x}_1 + \ddot{x}_2 &= -\dfrac{k}{m}(x_1 + x_2) \\ x_1 - x_2 &= 0 \end{cases} \quad \Rightarrow \quad x_1(t) = x_2(t) = A_1 \cos(\omega_1 t + \phi_1).$$

When $x_1 + x_2 = 0$,

$$\begin{cases} x_1 + x_2 &= 0 \\ \ddot{x}_1 - \ddot{x}_2 &= -\dfrac{k+2K}{m}(x_1 - x_2). \end{cases} \quad \Rightarrow \quad x_1(t) = -x_2(t) = A_2 \cos(\omega_2 t + \phi_2).$$

At last the general solution (representing any dynamics of the system) is a linear combination of normal modes

$$\begin{cases} x_1(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ x_2(t) &= A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \end{cases},$$

where $\omega_1 = \sqrt{\frac{k}{m}}$, $\omega_2 = \sqrt{\frac{k+2K}{m}}$, and $A_{1,2}$, $\phi_{1,2}$ are arbitrary constants depending on initial condition.

Remark 1. An alternative approach is to solve for general solution first. Say the solution is found to be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = B_1 \begin{bmatrix} a \\ b \end{bmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{bmatrix} c \\ d \end{bmatrix} \cos(\omega_2 t + \phi_2),$$

then one can eliminate one mode out of two to get normal coordinates $d\,x_1 - c\,x_2$ and $b\,x_1 - a\,x_2$.

Remark 2. Yet another approach, based on the observation that the normal modes are the motion where both masses oscillates at the same frequency, is to look for solutions of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \cos(\omega t + \phi).$$

Upon plugging in, a quadratic equation for ω can be derived, solving two roots to yield all the normal modes.

This method in actually equivalent to using undetermined coefficients to decouple the system, as we shall see in the next problem.

2. **(Practice)** Find the governing equations, determine the normal modes, and write the general solution of the following spring-mass problem:

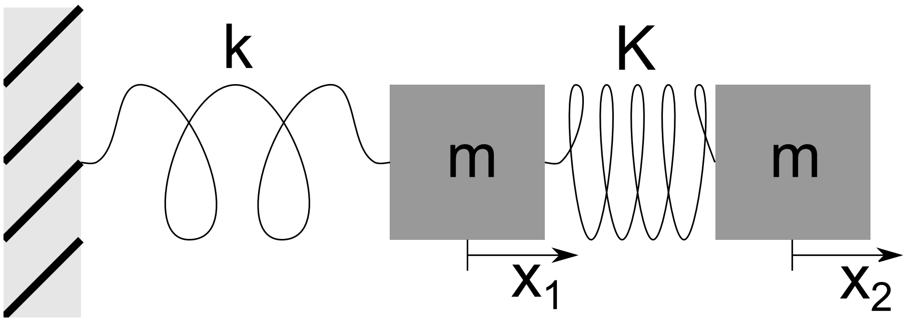

Figure 2. Two masses, two springs

Solution: Following the assumptions above in the demo, we can write the equations of motion

$$\begin{cases} \ddot{x}_1 = -\dfrac{k}{m} x_1 + \dfrac{K}{m}(x_2 - x_1), \\ \ddot{x}_2 = -\dfrac{K}{m}(x_2 - x_1). \end{cases}$$

The difference is a missing force in the second equation, which breaks symmetry and the normal coordinates are not immediately clear. But we may find them out through undetermined coefficients. Following the eqautions, for any $p, q \in \mathbb{R}$, we have

$$\begin{aligned} p\ddot{x}_1 + q\ddot{x}_2 &= \left(-p\frac{k+K}{m}x_1 + p\frac{K}{m}x_2\right) + \left(q\frac{K}{m}x_1 - q\frac{K}{m}x_2\right) \\ &= \frac{1}{m}[qK - p(k+K)]x_1 + \frac{1}{m}(pK - qK)x_2. \end{aligned}$$

To decouple the system, we need to make the equation for the combination autonomous, i.e.,

$$p\ddot{x}_1 + q\ddot{x}_2 = -\omega^2(px_1 + qx_2).$$

Therefore,

$$\frac{qK - p(k+K)}{p} = \frac{pK - qK}{q}.$$

Let $r=\frac{q}{p}$, then the condition above yields a quadratic equation for r (obviously $p,q\neq 0$)

$$\begin{aligned}
Kr-(k+K) &= \frac{1}{r}K-K \\
&\Updownarrow \\
Kr^2-kr-K &= 0 \\
&\Updownarrow \\
r^2-\frac{k}{K}r-1 &= 0 \\
&\Downarrow \\
r &= \frac{\frac{k}{K}\pm\sqrt{\left(\frac{k}{K}\right)^2+4}}{2} \\
&= \frac{1}{K}\left[\frac{k}{2}\pm\sqrt{\left(\frac{k}{2}\right)^2+K^2}\right].
\end{aligned}$$

Denote the two roots as $r_\pm$, then

$$\ddot{x}_1+r_\pm\ddot{x}_2 = -\omega_\pm^2(x_1+r_\pm x_2),$$

where the frequencies satisfy

$$\begin{aligned}
-\omega_\pm^2 &= \frac{1}{m}[r_\pm K-(k+K)] \\
\omega_\pm &= \sqrt{-\frac{1}{m}[r_\pm K-(k+K)]} \\
&= \sqrt{-\frac{1}{m}\left[\frac{k}{2}\pm\sqrt{\left(\frac{k}{2}\right)^2+K^2}-(k+K)\right]} \\
&= \sqrt{\frac{1}{m}\left[K+\frac{k}{2}\mp\sqrt{\left(\frac{k}{2}\right)^2+K^2}\right]}.
\end{aligned}$$

Now we may write the normal modes as $A_\pm\cos(\omega_\pm t+\phi_\pm)$, with $A_\pm,\phi_\pm$ dependent on initial values, corresponding to normal coordinates $x_1+r_\pm x_2$. Thus the general solution can be obtained from

$$\begin{cases} x_1+r_+x_2 = A_+\cos(\omega_+t+\phi_+) \\ x_1+r_-x_2 = A_-\cos(\omega_-t+\phi_-) \end{cases}.$$

Therefore,

$$\begin{aligned}
\left(\frac{1}{r_+}-\frac{1}{r_-}\right)x_1 &= \frac{A_+}{r_+}\cos(\omega_+t+\phi_+)-\frac{A_-}{r_-}\cos(\omega_-t+\phi_-) \\
x_1 &= \frac{1}{r_--r_+}[r_-A_+\cos(\omega_+t+\phi_+)-r_+A_-\cos(\omega_-t+\phi_-)] \\
&= A_+\frac{r_-}{r_--r_+}\cos(\omega_+t+\phi_+)-A_-\frac{r_+}{r_--r_+}\cos(\omega_-t+\phi_-).
\end{aligned}$$

And

$$(r_+ - r_-)\, x_2 \;=\; A_+ \cos(\omega_+ t + \phi_+) - A_- \cos(\omega_- t + \phi_-)$$

$$x_2 \;=\; A_+ \frac{1}{r_+ - r_-} \cos(\omega_+ t + \phi_+) - A_- \frac{1}{r_+ - r_-} \cos(\omega_- t + \phi_-).$$

Remark 3. As mentioned before, we may also find the normal modes by looking for particular solutions of the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \cos(\omega\, t + \phi).$$

Then

$$\begin{aligned} \ddot{x}_1 &= -u\, \omega^2 \cos(\omega\, t + \phi), \\ \ddot{x}_2 &= -v\, \omega^2 \cos(\omega\, t + \phi). \end{aligned}$$

Comparing with the right hand side of the equations of motion, we have

$$\begin{aligned} -u\, \omega^2 \cos(\omega\, t + \phi) &= -\frac{k}{m} u \cos(\omega\, t + \phi) + \frac{K}{m}(v - u) \cos(\omega\, t + \phi), \\ -v\, \omega^2 \cos(\omega\, t + \phi) &= -\frac{K}{m} (v - u) \cos(\omega\, t + \phi). \end{aligned}$$

Therefore,

$$\begin{aligned} -u\, \omega^2 &= -\frac{k}{m} u + \frac{K}{m}(v - u), \\ -v\, \omega^2 &= -\frac{K}{m} (v - u). \end{aligned}$$

Dividing the two eqautions, and denote $s = \frac{u}{v}$, we have an equation for s

$$\begin{aligned} s &= \frac{-\frac{k}{K} s + 1 - s}{s - 1} \\ s^2 + \frac{k}{K} s - 1 &= 0. \end{aligned}$$

Solving for the roots $s_\pm$ and one can get the same results via $\omega^2 = \frac{K}{m}(1 - s)$.

3. **(Practice)** Find the governing equations, determine the normal modes, and write the general solution of the following spring-mass problem:

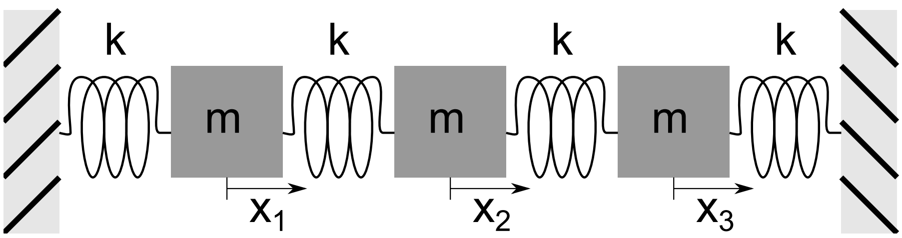

Figure 3. Three masses, four springs

Solution: First we write the equations of motion just like above

$$\begin{cases} \ddot{x}_1 &= -\frac{k}{m}x_1 + \frac{k}{m}(x_2 - x_1) \\ \ddot{x}_2 &= -\frac{k}{m}(x_2 - x_1) + \frac{k}{m}(x_3 - x_2) \\ \ddot{x}_3 &= -\frac{k}{m}(x_3 - x_2) + \frac{k}{m}(-x_3) \end{cases}.$$

Rewriting the system into vector form

$$\ddot{\boldsymbol{x}} = \frac{k}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\boldsymbol{x}$$

Denote the matrix above as A. The normal modes can be found by assuming each mass moves with a common frequency (but with its own amplitude and phase). Then $\ddot{x}_i = -\omega^2 x_i$, $\forall i$, i.e.

$$\ddot{\boldsymbol{x}} = \frac{k}{m}A\boldsymbol{x} = -\omega^2 I\boldsymbol{x}.$$

The equation above should have nontrivial solution, meaning that $-\omega^2$ is eigenvalue of $\frac{k}{m}A$. Since three eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -2 - \sqrt{2}, \\ \lambda_2 &= -2, \\ \lambda_3 &= -2 + \sqrt{2}, \end{aligned}$$

with corresponding eigenvectors

$$\begin{aligned} v_1 &= \frac{1}{2}[1, -\sqrt{2}, 1]^T, \\ v_2 &= \frac{1}{\sqrt{2}}[-1, 0, 1]^T, \\ v_3 &= \frac{1}{2}[1, \sqrt{2}, 1]^T, \end{aligned}$$

we have $\omega_i^2 = -\frac{k}{m}\lambda_i$, thus

$$\begin{aligned} \omega_1 &= \sqrt{(2 + \sqrt{2})\frac{k}{m}}, \\ \omega_2 &= \sqrt{\frac{2k}{m}}, \\ \omega_3 &= \sqrt{(2 - \sqrt{2})\frac{k}{m}}. \end{aligned}$$

Therefore, the normal modes are $A_i \cos(\omega_i t + \phi_i)$.

Denote $P=[v_1,v_2,v_3], \Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, then $A=P\Lambda P^T$, and

$$\begin{aligned} \ddot{\boldsymbol{x}} &= \frac{k}{m}A\boldsymbol{x} \\ &\Downarrow \\ P^T\ddot{\boldsymbol{x}} &= \frac{k}{m}\Lambda(P^T\boldsymbol{x}), \end{aligned}$$

in other words, the normal coordinates are $P^T\boldsymbol{x}$, i.e.

$$\begin{aligned} P^T\boldsymbol{x} &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}\boldsymbol{x} \\ &= \begin{bmatrix} \frac{1}{2}x_1-\frac{1}{\sqrt{2}}x_2+\frac{1}{2}x_3 \\ -\frac{1}{\sqrt{2}}x_1+\frac{1}{\sqrt{2}}x_3 \\ \frac{1}{2}x_1+\frac{1}{\sqrt{2}}x_2+\frac{1}{2}x_3 \end{bmatrix}. \end{aligned}$$

At last, the general solution is obtained through the relation $P^T\boldsymbol{x}=[A_i\cos(\omega_i t+\phi_i)]$,

$$\begin{aligned} \boldsymbol{x} &= P\,[A_i\cos(\omega_i t+\phi_i)] \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} A_1\cos(\omega_1 t+\phi_1) \\ A_2\cos(\omega_2 t+\phi_2) \\ A_3\cos(\omega_3 t+\phi_3) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}A_1\cos(\omega_1 t+\phi_1)-\frac{1}{\sqrt{2}}A_2\cos(\omega_2 t+\phi_2)+\frac{1}{2}A_3\cos(\omega_3 t+\phi_3) \\ -\frac{1}{\sqrt{2}}A_1\cos(\omega_1 t+\phi_1)+\frac{1}{\sqrt{2}}A_3\cos(\omega_3 t+\phi_3) \\ \frac{1}{2}A_1\cos(\omega_1 t+\phi_1)+\frac{1}{\sqrt{2}}A_2\cos(\omega_2 t+\phi_2)+\frac{1}{2}A_3\cos(\omega_3 t+\phi_3) \end{bmatrix}. \end{aligned}$$