Solutions to "Week 08 Worksheet"*

Systems of Equations

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Math 2352–Spring 2016

1. **(Demonstration)** Find the general solution and sketch the phase space diagrams of

   a) $\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1$

   b) $\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = -x_1 + x_2$

   c) $\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -x_1 - 3\,x_2$

   Solution:

   a) The system is first casted into matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   Then just like the scalar case,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{At}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix},$$

   where the matrix exponential can be calculated through its eigenvalue decomposition

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Then

$$\begin{aligned} e^{At} &= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{t} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{-t}+e^{t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{bmatrix}. \end{aligned}$$

   Therefore,

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= e^{At}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} x_1(0)\,(e^{-t}+e^{t}) + x_2(0)\,(e^{t}-e^{-t}) \\ x_1(0)\,(e^{t}-e^{-t}) + x_2(0)\,(e^{-t}+e^{t}) \end{bmatrix} \\ &= C_1\begin{bmatrix} e^{-t}+e^{t} \\ e^{t}-e^{-t} \end{bmatrix} + C_2\begin{bmatrix} e^{t}-e^{-t} \\ e^{-t}+e^{t} \end{bmatrix}, \end{aligned}$$

*. This document has been written using the GNU TeXmacs text editor (see www.texmacs.org).

where $C_i = \frac{1}{2}x_i(0)$ are constants that are dependent on initial conditions.

Or equivalently, the solution is simpler to be written in terms of hyperbolic functions

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(0)\begin{bmatrix} \cosh t \\ \sinh t \end{bmatrix} + x_2(0)\begin{bmatrix} \sinh t \\ \cosh t \end{bmatrix}.$$

To sketch the phase space diagram with crucial qualitative features, the key is to identify stationary points, analyse their stability, and consider asymptotic behavior near/far from those points.

The phase space diagram (unique stationary point at origin as a saddle point, orthogonal to both x and y axis, parallel to $x_1 = \pm x_2$ when $t \longrightarrow \infty$):

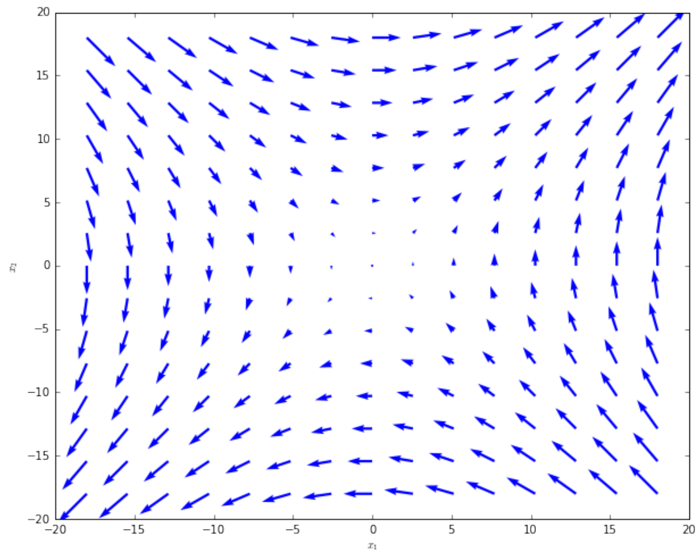

Figure 1.

Remark 1. For a 2×2 matrix, there is a shortcut to write its characteristic equation:

$$\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0.$$

More generally, for any matrix $\mathrm{tr}(A)$ is the sum of all eigenvalues, while $\det(A)$ is the product of them.

b) Here we employ another perspective of this kind of problems. The equation is to be solved via Laplace Transform method.

The first step is of course to apply Laplace transform on both equations

$$\begin{aligned} s\,\mathfrak{L}\{x_1\} - x_1(0) &= \mathfrak{L}\{x_1\} + \mathfrak{L}\{x_2\}, \\ s\,\mathfrak{L}\{x_2\} - x_2(0) &= -\mathfrak{L}\{x_1\} + \mathfrak{L}\{x_2\}. \end{aligned}$$

Now the system becomes an algebraic linear system,

$$\begin{cases} (s-1)\,\mathfrak{L}\{x_1\} - \mathfrak{L}\{x_2\} = x_1(0) \\ \mathfrak{L}\{x_1\} + (s-1)\,\mathfrak{L}\{x_2\} = x_2(0) \end{cases}.$$

Solving this system gives

$$\begin{aligned}
\mathfrak{L}\{x_1\} &= \frac{(s-1)\,x_1(0)+x_2(0)}{(s-1)^2+1},\\
\mathfrak{L}\{x_2\} &= \frac{(s-1)\,x_2(0)-x_1(0)}{(s-1)^2+1}.
\end{aligned}$$

At last we apply the inverse Laplace transform on the solutions,

$$\begin{aligned}
x_1(t) &= \mathfrak{L}^{-1}\left\{\frac{(s-1)\,x_1(0)+x_2(0)}{(s-1)^2+1}\right\}\\
&= x_1(0)\,\mathfrak{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\}+x_2(0)\,\mathfrak{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}\\
&= x_1(0)\,e^t\cos t+x_2(0)\,e^t\sin t.\\
x_2(t) &= \mathfrak{L}^{-1}\left\{\frac{(s-1)\,x_2(0)-x_1(0)}{(s-1)^2+1}\right\}\\
&= x_2(0)\mathfrak{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\}-x_1(0)\mathfrak{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}\\
&= x_2(0)\,e^t\cos t-x_1(0)\,e^t\sin t.
\end{aligned}$$

Phase space diagram (unique stationary point at origin as an unstable point):

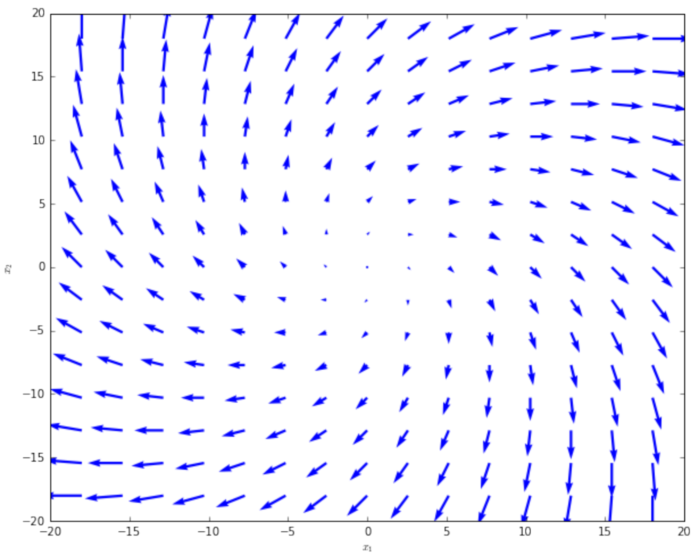

Figure 2.

Remark 2. This method allows us to solve the nonhomogeneous linear systems too. What's more, it can also handle discontinuous functions to some extent.

c) We solve this problem by pointing out the relation between ODE systems and higher-order ODEs. Apply $\frac{d}{\mathrm{dt}}$ on both sides of the first equation, we have

$$\begin{aligned} x_1'' &= -x_1' + x_2'. \\ &\Downarrow \\ x_2' &= x_1'' + x_1'. \end{aligned}$$

Plugging into the second equation, we have

$$\begin{aligned} x_1'' + x_1' &= -x_1 - 3\,x_2. \\ &\Downarrow \\ x_2 &= -\frac{1}{3}(x_1'' + x_1' + x_1). \end{aligned}$$

Substituting back to the first equation, we have eliminated x_2 from it by adding second order terms

$$\begin{aligned} x_1' &= -x_1 - \frac{1}{3}(x_1'' + x_1' + x_1). \\ &\Downarrow \\ x_1'' + 4x_1' + 4x_1 &= 0. \end{aligned}$$

And this happens to be a second order homogeneous ODE with constant coefficients, which solves

$$x_1(t) = C_1\,e^{-2t} + C_2\,t\,e^{-2t}.$$

Therefore,

$$\begin{aligned} x_1' &= -2C_1\,e^{-2t} + C_2\,e^{-2t} - 2C_2\,t\,e^{-2t} \\ &= (-2C_1 + C_2)e^{-2t} - 2C_2\,t\,e^{-2t}. \end{aligned}$$

$$\begin{aligned} x_1'' &= -2(-2C_1 + C_2)e^{-2t} - 2C_2\,e^{-2t} + 4C_2\,t\,e^{-2t} \\ &= (4C_1 - 4C_2)e^{-2t} + 4C_2\,t\,e^{-2t}. \end{aligned}$$

Thus

$$\begin{aligned} x_2 &= -\frac{1}{3}(x_1'' + x_1' + x_1) \\ &= -\frac{1}{3}(C_1 - 2C_1 + C_2 + 4C_1 - 4C_2)\,e^{-2t} \\ &\quad -\frac{1}{3}(C_2 - 2C_2 + 4C_2)\,t\,e^{-2t} \\ &= -(C_1 - C_2)\,e^{-2t} - C_2\,t\,e^{-2t} \\ &= C_1(-e^{-2t}) - C_2\,e^{-2t}\,(t-1). \end{aligned}$$

In summary, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\,e^{-2t}\begin{bmatrix} t \\ 1-t \end{bmatrix}.$$

Phase space diagram (unique stationary point at origin as a stable point):

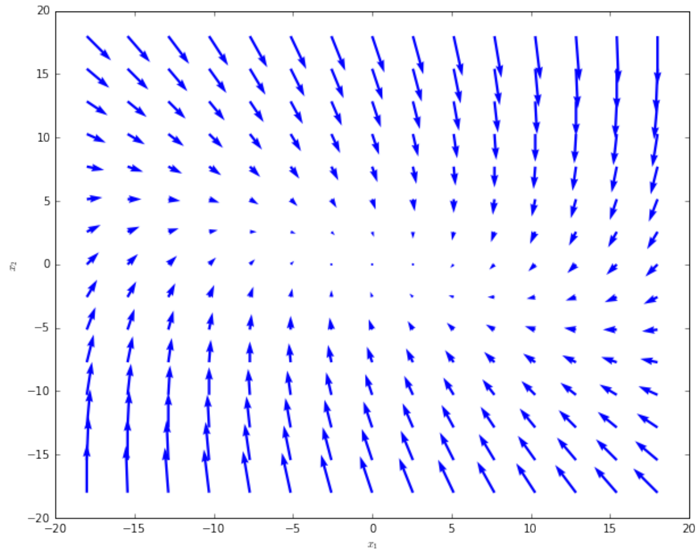

Figure 3.

Remark 3. The characteristic equations obtained in all the methods above should be the same. As a consequence, the ways of handling special cases like repeated roots are closely related.

2. **(Practice)** Find the general solution and sketch the phase space diagrams of

a) $\dot{x}_1 = 7\,x_1 - 2\,x_2, \quad \dot{x}_2 = 2\,x_1 + 2\,x_2$

b) $\dot{x}_1 = -x_2, \quad \dot{x}_2 = -2\,x_1 - x_2$

Solution:

a) The general solution is

```
>>> from sympy import *
>>> from sympy.abc import t
>>> x = Function('x')
>>> y = Function('y')
>>> eq = (Eq(diff(x(t),t), 7*x(t) - 2*y(t)), Eq(diff(y(t),t), 2*x(t)
    + 2*y(t)))
>>> eq
```

```
(Eq(Derivative(x(t), t), 7*x(t) - 2*y(t)), Eq(Derivative(y(t), t),
2*x(t) + 2*y(t)))
```

```
>>> dsolve(eq)
```

```
[Eq(x(t), -2*C1*exp(3*t) - 2*C2*exp(6*t)), Eq(y(t), -4*C1*exp(3*t)
- C2*exp(6*t))]
```

i.e., $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2\,C_1 e^{3t} - 2\,C_2\,e^{6t} \\ -4C_1 e^{3t} - C_2\,e^{6t} \end{bmatrix}$. The phase space diagram is:

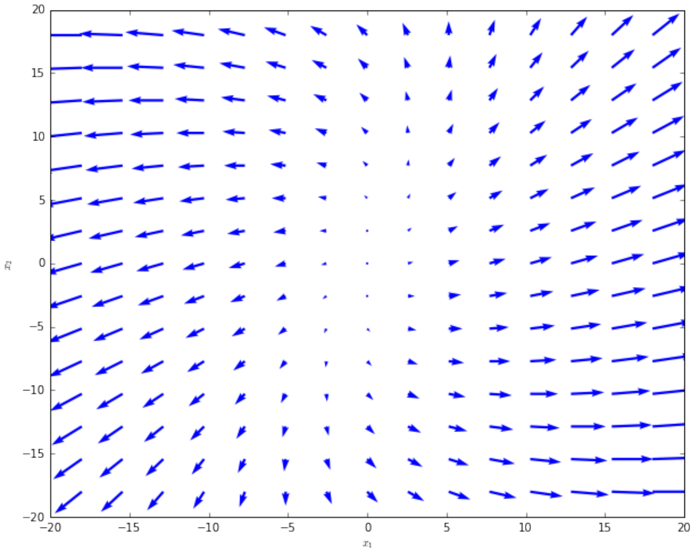

Figure 4.

b) The general solution is

```
>>> eq = (Eq(diff(x(t),t), -y(t)), Eq(diff(y(t),t), -2*x(t) - y(t)))
>>> dsolve(eq)
```

```
[Eq(x(t), -C1*exp(-2*t) - C2*exp(t)), Eq(y(t), -2*C1*exp(-2*t) +
C2*exp(t))]
```

i.e., $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -C_1 e^{-2t} - C_2\,e^{t} \\ -2C_1 e^{-2t} + C_2\,e^{t} \end{bmatrix}$. The phase space diagram is:

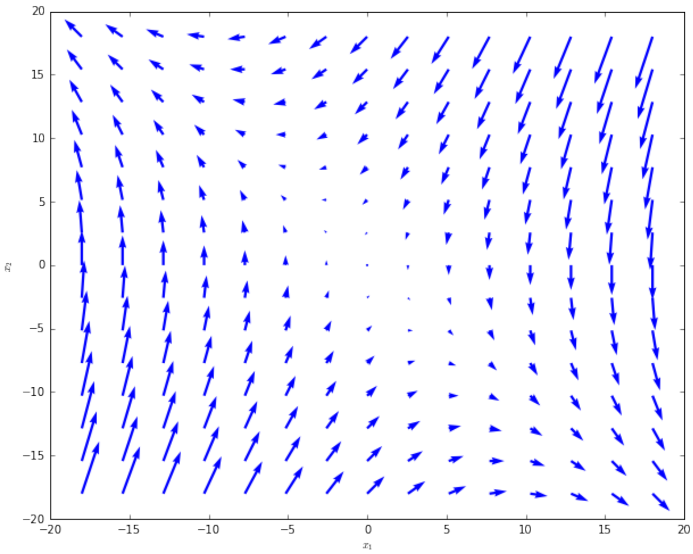

Figure 5.

3. **(Practice)** Find the general solution and sketch the phase space diagrams of

a) $\dot{x}_1 = x_1 - 2x_2, \quad \dot{x}_2 = x_1 + x_2$

b) $\dot{x}_1 = -x_1 - x_2, \quad \dot{x}_2 = x_1 - x_2$

Solution:

a) The general solution is

```
>>> eq = (Eq(diff(x(t),t), x(t)-2*y(t)), Eq(diff(y(t),t), x(t) +
    y(t)))
>>> dsolve(eq)
```

```
[Eq(x(t), -2*(C1*sin(sqrt(2)*t) + C2*cos(sqrt(2)*t))*exp(t)),
Eq(y(t), (sqrt(2)*C1*cos(sqrt(2)*t) -
sqrt(2)*C2*sin(sqrt(2)*t))*exp(t))]
```

i.e., $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^t \begin{bmatrix} -2\left(C_1 \sin(\sqrt{2}\,t) + C_2 \cos(\sqrt{2}\,t)\right) \\ \sqrt{2} C_1 \cos(\sqrt{2}\,t) - \sqrt{2} C_2 \sin(\sqrt{2}\,t) \end{bmatrix}$. The phase space diagram is:

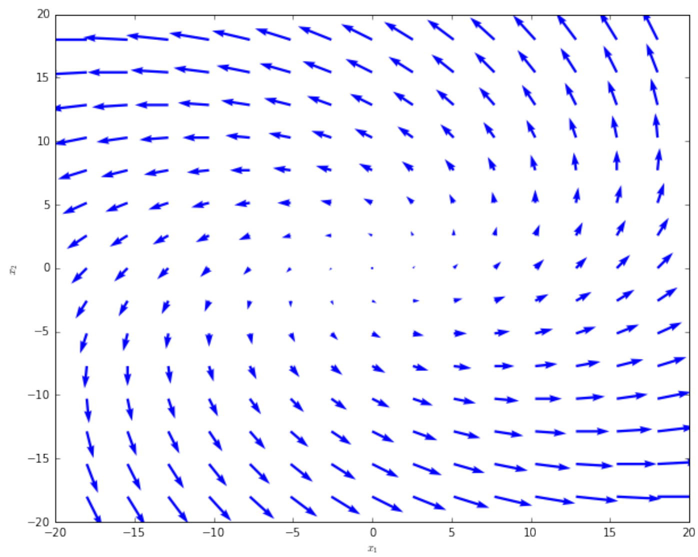

Figure 6.

b) The general solution is

```
>>> eq = (Eq(diff(x(t),t), -x(t)-y(t)), Eq(diff(y(t),t), x(t) -
    y(t)))
>>> dsolve(eq)
```

```
[Eq(x(t), -(C1*sin(t) + C2*cos(t))*exp(-t)), Eq(y(t), (C1*cos(t) -
C2*sin(t))*exp(-t))]
```

i.e., $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-t} \begin{bmatrix} -C_1 \sin(t) - C_2 \cos(t) \\ C_1 \cos(t) - C_2 \sin(t) \end{bmatrix}$. The phase space diagram is:

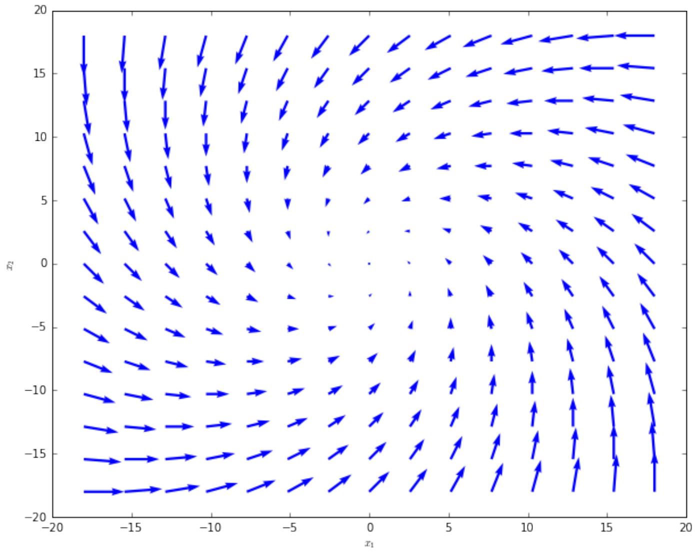

Figure 7.

4. **(Practice)** Find the general solution and sketch the phase space diagrams of

   1. $\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = -4\,x_1 - 3\,x_2$

   Solution: The general solution is

```
>>> eq = (Eq(diff(x(t),t), x(t)+y(t)), Eq(diff(y(t),t), -4*x(t) - 3*y(t)))
>>> dsolve(eq)
```

```
[Eq(x(t), (2*C1 + 2*C2*t + C2/2)*exp(-t)), Eq(y(t), (-4*C1 - 4*C2*t +
C2)*exp(-t))]
```

   i.e., $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 2\,C_1 + 2\,C_2\,t + \frac{C_2}{2} \\ -4C_1 - 4C_2\,t + C_2 \end{bmatrix}$. The phase space diagram is:

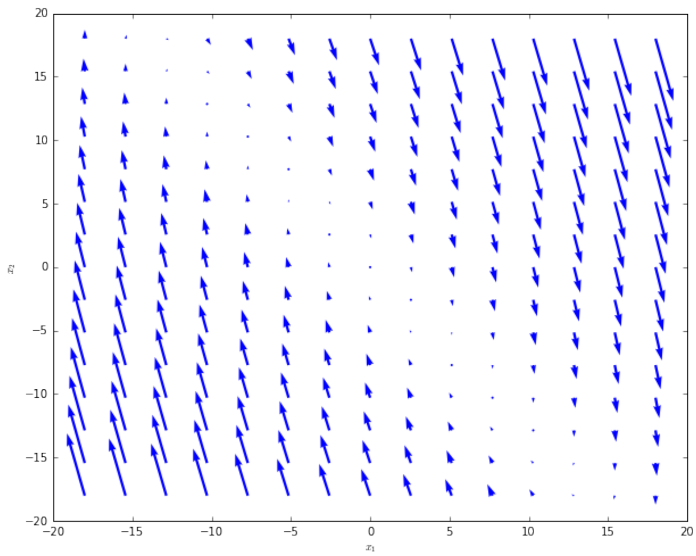

Figure 8.