Solutions to “Week 07 Worksheet”*

Cauchy-Euler Equations

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1. **(Demonstration)** Solve

   a) $x^2\, y'' + x\, y' - y = 0; \quad y(0) = 0,\ y(1) = 1$

   b) $x^2\, y'' - x\, y' + (1 + \pi^2 / 4)\, y = 0; \quad y(1) = 1,\ y(e) = e$

   c) $x^2\, y'' + 3\, x\, y' + y = 0; \quad y(1) = 1,\ y(e) = 1$

Solution: We provide three methods that are powerful enough to deal with general Cauchy-Euler equations. Each problem is solved using one of them respectively.

a) *[Solving through trial solution]*

We may directly look for solutions of the form $y(x) = x^r$. For this kind of functions, we have

$$\begin{aligned} x\, y' &= r\, y, \\ x^2 y'' &= r(r-1) y. \end{aligned}$$

Plugging into the ODE,

$$[r(r-1) + r - 1] y(x) = 0, \quad \forall \text{possible}\, x.$$

Then we have

$$\begin{aligned} (r+1)(r-1) &= 0 \\ r &= \pm 1. \end{aligned}$$

Now we have two independent solutions to the ODE, $y_1(x) = x$ and $y_2(x) = \frac{1}{x}$. Thanks to the linearity of the equation, any solution to it can be written as

$$y(x) = C_1\, x + C_2 \frac{1}{x}, \quad C_{1,2} \in \mathbb{R}.$$

Taking advantage of the boundary conditions, we have

$$\begin{aligned} C_2 &= 0, \\ C_1 &= 1. \end{aligned}$$

*. This document has been written using the GNU $\text{T}_{\text{E}}\text{X}_{\text{MACS}}$ text editor (see `www.texmacs.org`).

So the solution to the Cauchy problem should be

$$y(x) \;=\; x.$$

b) *[Solution through change of variables]*

Apply a change of variables by letting

$$t \;=\; \ln x.$$

And the solution is now a function dependent on t,

$$y(x) = y(e^t) \;\triangleq\; \Phi(t).$$

Differentiating,

$$\begin{aligned} y'(x) &= \Phi'(t)\,\frac{1}{x}, \\ y''(x) &= \Phi''(t)\frac{1}{x^2} + \Phi'(t)\left(-\frac{1}{x^2}\right) \\ &= [\Phi''(t) - \Phi'(t)]\frac{1}{x^2}. \end{aligned}$$

Substitute into the ODE, we have

$$\Phi''(t) - 2\Phi'(t) + (1 + \pi^2/4)\,\Phi(t) \;=\; 0.$$

And the boundary conditions

$$\begin{aligned} 1 = y(1) &= \Phi(0), \\ e = y(e) &= \Phi(1) \end{aligned}$$

Now the equation is of constant coefficients and can be easily solved using its characteristic polynomial

$$\begin{aligned} r^2 - 2r + (1 + \pi^2/4) &= 0 \\ r &= 1 \pm i\frac{\pi}{2}. \end{aligned}$$

Then

$$\Phi(t) \;=\; C_1 e^{\left(1 + i\frac{\pi}{2}\right)t} + C_2 e^{\left(1 - i\frac{\pi}{2}\right)t}, \quad C_{1,2} \in \mathbb{C}.$$

Making use of the boundary conditions

$$\begin{aligned} 1 = \Phi(0) &= C_1 + C_2, \\ e = \Phi(1) &= C_1 e^{\left(1 + i\frac{\pi}{2}\right)} + C_2 e^{\left(1 - i\frac{\pi}{2}\right)}. \end{aligned}$$

The second equation is equivalen to

$$C_1 - C_2 \;=\; -i.$$

So we have $C_1 = \frac{1-i}{2}$, $C_2 = \frac{1+i}{2}$, thus

$$
\begin{aligned}
\Phi(t) &= C_1 e^{(1+i\frac{\pi}{2})t} + C_2 e^{(1-i\frac{\pi}{2})t} \\
&= e^t \left[C_1 e^{i\frac{\pi}{2}t} + C_2 e^{-i\frac{\pi}{2}t}\right] \\
&= e^t \frac{1}{2}\Big[(C_1+C_2)\big(e^{i\frac{\pi}{2}t} + e^{-i\frac{\pi}{2}t}\big) + (C_1-C_2)\big(e^{i\frac{\pi}{2}t} - e^{-i\frac{\pi}{2}t}\big)\Big] \\
&= e^t \left[\cos\Big(\frac{\pi}{2}t\Big) + \sin\Big(\frac{\pi}{2}t\Big)\right].
\end{aligned}
$$

Transform back to $x = e^t$, we have

$$
y(x) = x\left[\cos\Big(\frac{\pi}{2}\ln x\Big) + \sin\Big(\frac{\pi}{2}\ln x\Big)\right], \quad x > 0.
$$

c) *[Series solution]*

As seen in a), solutions to Cauchy-Euler equations may contain functions like $\frac{1}{x}$ that cannot be expanded at $x=0$. However, series solutions may still be found by the so-called Frobenius method (Laurent series), simply by assuming a solution of the form

$$
\begin{aligned}
y(x) &= x^r \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n x^{n+r}.
\end{aligned}
$$

In addition, we assume that $a_0 \neq 0$, this does not affect generality since we can always adjust r to acheive that for any nonzero series. By doing so we can fix r to be certain values (removing null space). Then we have

$$
\begin{aligned}
x\,y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r}, \\
x^2 y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r}.
\end{aligned}
$$

Plugging into the ODE, we have

$$
\sum_{n=0}^{\infty} [(n+r)(n+r-1)a_n + 3(n+r)\,a_n + a_n] x^{n+r} = 0, \quad \forall \text{possible}\, x.
$$

Therefore,

$$
(n+r)(n+r-1)a_n + 3(n+r)\,a_n + a_n = 0, \quad \forall n \in \mathbb{N}.
$$

Setting $n=0$, and noting $a_0 \neq 0$, we have a quadratic equation for r,

$$
\begin{aligned}
r^2 + 2r + 1 &= 0 \\
r &= -1.
\end{aligned}
$$

Plugging into the recurrence relation,

$$n^2 a_n \;=\; 0, \quad \forall n \geqslant 1.$$

Note that this is a case where we have repeated roots. One solution is thus what we started out looking for

$$\begin{aligned} y_1(x) &= x^{-1} \sum_{n=0}^{\infty} a_n x^n \\ &= \frac{a_0}{x}. \end{aligned}$$

While the other one is of the form

$$y_2(x) \;=\; y_1(x)\log x + x^r \sum_{n=1}^{\infty} a_n x^n.$$

Note that the term $n=0$ is ommitted as it would just give a multiple of $y_1(x)$. Therefore,

$$y_2(x) \;=\; a_0 \frac{\log x}{x}.$$

Now, the general solution is

$$y(x) \;=\; C_1 \frac{1}{x} + C_2 \frac{\log x}{x}, \quad C_{1,2} \in \mathbb{R}.$$

Using the boundary conditions

$$\begin{aligned} 1 = y(1) &= C_1 \\ 1 = y(e) &= \frac{C_1}{e} + \frac{C_2}{e} \end{aligned}$$

we have $C_1 = 1$, $C_2 = e - 1$. So

$$y(x) \;=\; \frac{1}{x} + (e-1)\frac{\log x}{x}.$$

Remark 1. The "characteristic equations" w.r.t. r from all three methods above are the same.

Remark 2. When there are repeated root, one should look for

i. First method: $x^r \log x$, $x^r (\log x)^2, \ldots$

ii. Second method: $t\, e^{rt}$, $t^2 e^{rt}$, ...

iii. Third method: $y_2(x) \;=\; y_1(x)\log x + x^r \sum_{n=1}^{\infty} a_n x^n$ ...

Remark 3. The series solution method is the most powerful one, which can deal with more general equations of the form

$$x^2\, y'' + x\, p(x)\, y' - q(x)\, y \;=\; 0$$

with $p(x)$, $q(x)$ expandable into Taylor series at $x=0$. However, the powerfulness comes with a price. In addition to dealing with repeated roots, one also need to deal with another special case where the two roots are differed by an integer, $r_1 - r_2 = N$, $N \in \mathbb{Z}^+$ similarly to the repeated-root case.

2. **(Practice)** Solve

   a) $x^2\, y'' - 2\, x\, y' + 2\, y = 0, \quad y'(0) = 1,\ y(1) = 0$

   b) $2\, x^2\, y'' - x\, y' + y = 0, \quad y(1) = 0,\ y(4) = 1$

   Solution:

   a) The general solution is

```
>>> from sympy import Function, dsolve, diff
>>> from sympy.abc import x
>>> y = Function('y')
>>> solu = dsolve(diff(y(x),x,x)*x*x - diff(y(x),x)*x*2 + 2*y(x),
    y(x), hint='nth_linear_euler_eq_homogeneous')
>>> solu
```

```
Eq(y(x), x*(C1 + C2*x))
```

   i.e., $y(x) = C_1 x + C_2\, x^2$. Applying boundary conditions, we have

$$y(x) \;=\; -(x-1)x.$$

   b) The general solution is

```
>>> solu = dsolve(diff(y(x),x,x)*2*x*x - diff(y(x),x)*x + y(x), y(x),
    hint='nth_linear_euler_eq_homogeneous')
>>> solu
```

```
Eq(y(x), C1*sqrt(x) + C2*x)
```

   i.e., $y(x) = C_1 x^{\frac{1}{s}} + C_2\, x$. Applying boundary conditions, we have

$$y(x) \;=\; \frac{1}{2}(x - \sqrt{x}).$$

3. **(Practice)** Solve

   a) $x^2\, y'' - x\, y' + (1+\pi^2)\, y = 0, \quad y(1) = 1,\ y(\sqrt{e}) = \sqrt{e}$

   b) $x^2\, y'' + 3\, x\, y' + (1+\pi^2)\, y = 0, \quad y(1) = 1,\ y(\sqrt{e}) = \sqrt{e}$

   Solution:

   a) The general solution is

```
>>> from sympy import pi
>>> solu = dsolve(diff(y(x),x,x)*x*x - diff(y(x),x)*x +
    (1+pi*pi)*y(x), y(x), hint='nth_linear_euler_eq_homogeneous')
```

```
>>> solu
   Eq(y(x), x*(C1*sin(pi*log(x)) + C2*cos(pi*log(x))))
```

i.e., $y(x) = C_1 x \sin(\pi \log x) + C_2\, x \cos(\pi \log x)$. Applying boundary conditions, we have

$$y(x) \;=\; x \sin(\pi \log x) + x \cos(\pi \log x).$$

b) The general solution is

```
>>> solu = dsolve(diff(y(x),x,x)*x*x + diff(y(x),x)*3*x +
    (1+pi*pi)*y(x), y(x), hint='nth_linear_euler_eq_homogeneous')
>>> solu
   Eq(y(x), (C1*sin(pi*log(x)) + C2*cos(pi*log(x)))/x)
```

i.e., $y(x) = C_1 \frac{1}{x} \sin(\pi \log x) + C_2\, \frac{1}{x} \cos(\pi \log x)$. Applying boundary conditions, we have

$$y(x) \;=\; \frac{e}{x} \sin(\pi \log x) + \frac{e}{x} \cos(\pi \log x).$$

4. **(Practice)** Solve

a) $x^2\, y'' - x\, y' + y = 0, \quad y(1) = 1,\; y'(1) = 0$

b) $4\, x^2\, y'' + y = 0, \quad y(1) = 1,\; y(e) = 0$

Solution:

a) The general solution is

```
>>> solu = dsolve(diff(y(x),x,x)*x*x - diff(y(x),x)*x + y(x), y(x),
    hint='nth_linear_euler_eq_homogeneous')
>>> solu
   Eq(y(x), x*(C1 + C2*log(x)))
```

i.e., $y(x) = C_1 x + C_2\, x \log x$. Applying initial conditions, we have

$$y(x) \;=\; x - x \log x.$$

b) The general solution is

```
>>> solu = dsolve(diff(y(x),x,x)*4*x*x + y(x), y(x),
    hint='nth_linear_euler_eq_homogeneous')
>>> solu
   Eq(y(x), sqrt(x)*(C1 + C2*log(x)))
```

i.e., $y(x) = C_1 x^{\frac{1}{2}} + C_2\, x^{\frac{1}{2}} \log x$. Applying boundary conditions, we have

$$y(x) \;=\; -\sqrt{x}\,(\log x - 1).$$