

## Solution to Worksheet 6: Series solutions

## 1. (Demonstration)

2. (Practice) Find two independent power series solutions to the following differential equation, where the highest power of  $x$  to be computed is specified:

(a)  $y'' + xy' - y = 0 \quad (x^6)$

With

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the differential equation becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

We shift the first term to  $x^n$  by shifting the exponent up by 2, i.e.,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} + (n-1)a_n)x^n = 0,$$

Which implies,

$$a_{n+2} = \frac{1-n}{(n+1)(n+2)} a_n.$$

Since  $a_3 = 0$ , we find the sequence with odd number

$$a_3 = a_5 = a_7 = \cdots = 0,$$

and sequence starting with  $a_0$ ,

$$\begin{aligned} a_0, \\ a_2 &= \frac{1}{2}a_0, \\ a_4 &= -\frac{1}{3 \cdot 4}a_2 = -\frac{1}{4!}a_0, \\ a_6 &= \frac{3}{6!}a_0. \end{aligned}$$

Therefore the solution for  $y = y(x)$  could be written as

$$y(x) = a_0 \left( 1 + \frac{1}{2}x^2 - \frac{1}{4!}x^4 + \frac{3}{6!}x^6 + \cdots \right) + a_1 x.$$

(b)  $y'' + y' + xy = 0 \quad (x^5)$

With

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the differential equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^{n-1} + \sum_{n=0}^{\infty} a_nx^{n+1} = 0,$$

or

$$\sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} + (n+2)a_{n+2} + a_n]x^{n+1} = 0, \quad 2a_2 + a_1 = 0,$$

Therefore, we obtain the sequence

$$a_{n+3} = -\frac{(n+2)a_{n+2} + a_n}{(n+2)(n+3)}, \quad 2a_2 + a_1 = 0,$$

or

$$\begin{aligned} a_0, \\ a_1, \\ a_2 &= -\frac{1}{2}a_1, \\ a_3 &= -\frac{2a_2 + a_0}{6} = \frac{a_1 - a_0}{6}, \\ a_4 &= -\frac{3a_3 + a_1}{4 \cdot 3} = \frac{a_0 - 3a_1}{24}, \\ a_5 &= -\frac{4a_4 + a_2}{5 \cdot 4} = \frac{6a_1 - a_0}{120}. \end{aligned}$$

Therefore the solution for  $y = y(x)$  could be written as

$$y(x) = a_0 + a_1x - \frac{1}{2}a_1x^2 + \frac{a_1 - a_0}{6}x^3 + \frac{a_0 - 3a_1}{24}x^4 + \frac{6a_1 - a_0}{120}x^5 + \dots$$

3. **(Practice)** The Hermite equation is given by

$$y'' - 2xy' + 2\lambda y = 0,$$

where  $\lambda$  is a constant.

(a) Find the first three terms in each of two independent power series solutions.

With

$$y(x) = \sum_{n=0}^{\infty} a_nx^n,$$

the differential equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2 \sum_{n=0}^{\infty} na_nx^n + 2\lambda \sum_{n=0}^{\infty} a_nx^n = 0,$$

from which we can find

$$a_{n+2} = \frac{2(n-\lambda)}{(n+1)(n+2)}a_n.$$

Therefore, the sequence starting with  $a_0$

$$\begin{aligned} a_0, \\ a_2 &= -\lambda a_0, \\ a_4 &= \frac{\lambda(\lambda-2)}{6}a_0, \end{aligned}$$

and the sequence starting with  $a_1$

$$\begin{aligned} a_1, \\ a_3 &= \frac{1-\lambda}{3}a_1, \\ a_5 &= \frac{(\lambda-1)(\lambda-3)}{30}a_1. \end{aligned}$$

(b) If  $\lambda = n$  is a nonnegative integer, then one of the power series solutions will become a polynomial. Find the polynomial solutions for  $n = 0, 1, 2, 3$ .

- i.  $\lambda = 0 : y(x) = a_0$
- ii.  $\lambda = 1 : y(x) = a_1x$
- iii.  $\lambda = 2 : y(x) = a_0 - 2a_0x^2$
- iv.  $\lambda = 3 : y(x) = a_1x - \frac{2}{3}x^3$

(c) The Hermite polynomials  $H_n(x)$  are the polynomial solutions of the Hermite equation normalized so that the coefficient of  $x^n$  is  $2^n$ . Find  $H_0(x)$ ,  $H_1(x)$ ,  $H_2(x)$ ,  $H_3(x)$ .

To normalize the equation in (b) such that the coefficient of  $x^n$  is  $2^n$ , we should take  $a_0$  equals 1 and  $-2$  respectively when  $\lambda = 0$  and 2, and take  $a_1$  is equal to 2 and  $-12$  respectively when  $\lambda = 1$  and 3. Therefore, we obtain the Hermite equation

- i.  $\lambda = 0 : y(x) = 1$
- ii.  $\lambda = 1 : y(x) = 2x$
- iii.  $\lambda = 2 : y(x) = 4x^2 - 2$
- iv.  $\lambda = 3 : y(x) = 8x^3 - 12x$