Math 2352—Spring 2016

Solution to Worksheet 6: Series solutions

1. **(Demonstration)**

2. **(Practice)** Find two independent power series solutions to the following differential equation, where the highest power of x to be computed is specified:

   (a) $y'' + xy' - y = 0 \quad (x^6)$

   With
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$
   the differential equation becomes
$$\sum_{n=2} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$
   We shift the first term to x^n by shifting the exponent up by 2, i.e.,
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0,$$
   or
$$\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + (n-1)a_n)x^n = 0,$$
   Which implies,
$$a_{n+2} = \frac{1-n}{(n+1)(n+2)}a_n.$$
   Since $a_3 = 0$, we find the sequence with odd number
$$a_3 = a_5 = a_7 = \cdots = 0,$$
   and sequence starting with a_0,
$$\begin{aligned}
&a_0,\\
&a_2 = \frac{1}{2}a_0,\\
&a_4 = -\frac{1}{3\cdot 4}a_2 = -\frac{1}{4!}a_0,\\
&a_6 = \frac{3}{6!}a_0.
\end{aligned}$$
   Therefore the solution for $y = y(x)$ could be written as
$$y(x) = a_0(1 + \frac{1}{2}x^2 - \frac{1}{4!}x^4 + \frac{3}{6!}x^6 + \cdots) + a_1 x.$$

   (b) $y'' + y' + xy = 0 \quad (x^5)$

   With
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the differential equation becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

or

$$\sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3} + (n+2)a_{n+2} + a_n]x^{n+1} = 0, \quad 2a_2 + a_1 = 0,$$

Therefore, we obtain the sequence

$$a_{n+3} = -\frac{(n+2)a_{n+2} + a_n}{(n+2)(n+3)}, \quad 2a_2 + a_1 = 0,$$

or

$$\begin{aligned}
&a_0,\\
&a_1,\\
&a_2 = -\frac{1}{2}a_1,\\
&a_3 = -\frac{2a_2 + a_0}{6} = \frac{a_1 - a_0}{6},\\
&a_4 = -\frac{3a_3 + a_1}{4\cdot 3} = \frac{a_0 - 3a_1}{24},\\
&a_5 = -\frac{4a_4 + a_2}{5\cdot 4} = \frac{6a_1 - a_0}{120}.
\end{aligned}$$

Therefore the solution for $y = y(x)$ could be written as

$$y(x) = a_0 + a_1 x - \frac{1}{2}a_1 x^2 + \frac{a_1 - a_0}{6}x^3 + \frac{a_0 - 3a_1}{24}x^4 + \frac{6a_1 - a_0}{240}x^5 + \cdots.$$

3. **(Practice)** The Hermite equation is given by

$y'' - 2xy' + 2\lambda y = 0,$

where λ is a constant.

(a) Find the first three terms in each of two independent power series solutions.

With

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

the differential equation becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=0}^{\infty} n a_n x^n + 2\lambda \sum_{n=0}^{\infty} a_n x^n = 0,$$

from which we can find

$$a_{n+2} = \frac{2(n-\lambda)}{(n+1)(n+2)}a_n.$$

Therefore, the sequence starting with a_0

$$\begin{aligned}
&a_0,\\
&a_2 = -\lambda a_0,\\
&a_4 = \frac{\lambda(\lambda-2)}{6}a_0,
\end{aligned}$$

and the sequence starting with a_1

$$\begin{aligned} &a_1, \\ &a_3 = \frac{1-\lambda}{3} a_1, \\ &a_5 = \frac{(\lambda-1)(\lambda-3)}{30} a_1. \end{aligned}$$

(b) If $\lambda = n$ is a nonnegative integer, then one of the power series solutions will become a polynomial. Find the polynomial solutions for $n = 0, 1, 2, 3$.

i. $\lambda = 0 : y(x) = a_0$
ii. $\lambda = 1 : y(x) = a_1 x$
iii. $\lambda = 2 : y(x) = a_0 - 2a_0 x^2$
iv. $\lambda = 3 : y(x) = a_1 x - \frac{2}{3} x^3$

(c) The Hermite polynomials $H_n(x)$ are the polynomial solutions of the Hermite equation normalized so that the coefficient of x^n is 2^n. Find $H_0(x)$, $H_1(x)$, $H_2(x)$, $H_3(x)$.

To normalize the equation in (b) such that the coefficient of x^n is 2^n, we should take a_0 equals 1 and -2 respectively when $\lambda = 0$ and 2, and take a_1 is equal to 2 and -12 respectively when $\lambda = 1$ and 3. Therefore, we obtain the Hermite euqation

i. $\lambda = 0 : y(x) = 1$
ii. $\lambda = 1 : y(x) = 2x$
iii. $\lambda = 2 : y(x) = 4x^2 - 2$
iv. $\lambda = 3 : y(x) = 8x^3 - 12x$