Solution to Worksheet 5: The Laplace transform

1. **(Demonstration)**
2. **(Demonstration)**
3. **(Demonstration)**
4. **(Practice)** Solve the following inhomogeneous odes using the Laplace transform technique:

(a) $\ddot{x} + 2\dot{x} + 5x = e^{-2t}, \quad x(0) = 0,\ \dot{x}(0) = 0$

Taking the Laplace transform of both sides of the ode, we find

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 2[sX(s) - x(0)] + 5X(x) = \frac{1}{s+2},$$

or

$$\begin{aligned} X(s) &= \frac{1}{(s+2)(s^2+2s+5)} \\ &= \frac{1}{5}\frac{1}{s+2} - \frac{1}{5}\frac{s}{s^2+2s+5} \\ &= \frac{1}{5}\frac{1}{s+2} - \frac{1}{5}\frac{s-1}{(s+1)^2+2^2} + \frac{1}{5}\frac{s}{(s+1)^2+2^2}. \end{aligned}$$

By taking inverse Laplace transforms of the three terms separately, we obtain the solution

$$x(t) = \frac{1}{5}e^{-2t} - \frac{1}{5}e^{-t}\cos 2t + \frac{1}{10}e^{-t}\sin 2t.$$

(b) $\ddot{x} + 3\dot{x} + 2x = \begin{cases} 1-t & \text{if } 0 \le t < 1, \\ 0 & \text{if } t \ge 1; \end{cases} \quad x(0) = 0,\ \dot{x}(0) = 0$

There are 2 methods to obtain Laplace transform of right-hand side, let

$$g(x) = \begin{cases} 1-t & \text{if} \quad 0 \le t < 1, \\ 0 & \text{if} \quad t \ge 1. \end{cases}$$

① definition of Laplace transform, we have

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\,\mathrm{d}t \\ &= \int_0^1 e^{-st}(1-t)\,\mathrm{d}t \\ &= \frac{1}{s}(1-e^{-s}) - \frac{1}{s^2}(1-e^{-s}) + \frac{1}{s}e^{-s} \\ &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s}. \end{aligned}$$

② Otherwise $g(t)$ could be rewritten as

$$g(t) = (u_0 - u_1)(1-t) = u_0 - tu_0 + (t-1)u_1.$$

Then taking Laplace transfrom of the three terms separately we obtain

$$\mathcal{L}\{g(t)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}.$$

As we can see, the 2 methods have same result. Therefore, we find

$$s^2X(s) + 3sX(s) + 2X(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s^2}.$$

Or

$$X(s) = \frac{s - 1 + e^{-s}}{s^2(s+1)(s+2)}.$$

We write

$$X(s) = (e^{-s} - 1)H_1(s) + H_2(s),$$

where

$$\begin{aligned} H_1(s) &= \frac{1}{s^2(s+1)(s+2)} \\ &= \frac{1}{2s^2} + \frac{1}{s+1} - \frac{1}{4(s+2)} - \frac{3}{4s}; \\ H_2(s) &= \frac{1}{s(s+1)(s+2)} \\ &= \frac{1}{2(s+2)} + \frac{1}{2s} - \frac{1}{s+1}. \end{aligned}$$

Then we have

$$x(t) = u_1(t)h_1(t-1) - h_1(t) + h_2(t),$$

where

$$h_1(t) = \mathcal{L}^{-1}\{H_1(s)\} = -\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t};$$

$$h_2(t) = \mathcal{L}^{-1}\{H_2(s)\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

(c) $\ddot{x} + 2\dot{x} + x = \delta(t-1), \quad x(0) = 0,\ \dot{x}(0) = 1$

Taking Laplace transfrom we obtain

$$s^2X(s) - 1 + 2sX(s) + X(s) = e^{-s},$$

or

$$\begin{aligned} X(s) &= \frac{1 + e^{-s}}{s^2 + 2s + 1} \\ &= = \frac{1}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}. \end{aligned}$$

By taking inverse Laplace transfrom, we have

$$x(t) = te^{-t} + u_1(t-1)e^{1-t}.$$