Math 2352–Spring 2016

Solution to "Worksheet 4: Inhomogeneous odes"

1. **(Demonstration)**
2. **(Demonstration)**
3. **(Practice)** Find particular solutions:

   Remark *Notice particular solution is not unique, so the answers provided here are for reference only.*

   (a) $\ddot{x} + 3\dot{x} + 2x = e^{2t}$

   Ansatz
   $$x(t) = Ae^{2t}$$
   Upon substitution into the differential equation, we obtain
   $$4Ae^{2t} + 6Ae^{2t} + 2Ae^{2t} = e^{2t},$$
   from which we determine A = 1/12, therefore the particular solution is
   $$x_p = \frac{1}{12}e^{2t}$$

   (b) $\ddot{x} + 3\dot{x} + 2x = e^{-2t}$

   Since resonance(refer to page 42 of textbook), we take ansatz
   $$x(t) = Ate^{-2t},$$
   and take first and second derivatives, we obtain
   $$\dot{x} = Ae^{2t} - 2Ate^{-2t}, \qquad \ddot{x} = 4Ate^{-2t} - 4Ae^{-2t}.$$
   Substitution into the differential equation yields
   $$4Ate^{-2t} - 4Ae^{-2t} + 3(Ae^{2t} - 2Ae^{-2t}) + 2(Ate^{-2t}) = e^{-2t},$$
   from which A = -1, therefore the particular solution is
   $$x_p = -te^{-2t}$$

   (c) $\ddot{x} + 3\dot{x} + 2x = \sin 2t$

   Ansatz
   $$x(t) = A\sin 2t + B\cos 2t,$$
   Upon substitution into the differential equation, we obtain
   $$-4A\sin 2t - 4B\cos 2t + 3(2A\cos 2t - 2B\sin 2t) + 2(A\sin 2t + B\cos 2t) = \sin 2t,$$
   or regrouping terms,
   $$(-2A - 6B)\sin 2t + (6A - 2B)\cos 2t = \sin 2t, \tag{1}$$
   from which we obtain
   $$A = -\frac{1}{20} \qquad B = -\frac{3}{20}.$$
   Therefore the particular solution is
   $$x_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t.$$

(d) $\ddot{x} + 3\dot{x} + 2x = \cos 2t$

Because this question has same form as (c), we can use same ansatz. Replace $\sin 2t$ by $\cos 2t$ in (1) we obtain

$$(-2A - 6B)\sin 2t + (6A - 2B)\cos 2t = \cos 2t$$

and solving for A and B, we obtain

$$A = \frac{3}{20} \qquad B = -\frac{1}{20}.$$

Therefore the particular solution is

$$x_p = \frac{3}{20}\sin 2t - \frac{1}{20}\cos 2t$$

(e) $\ddot{x} + 3\dot{x} + 2x = 2t$

Ansatz

$$x(t) = At + B.$$

Upon substituting into the ode, we obtain

$$3A + 2(At + B) = 2t,$$

from which we can determine A = 1, B = -3/2. Therefore the particular solution is

$$x_p = t - \frac{3}{2}.$$

(f) $\ddot{x} + 3\dot{x} + 2x = t^2 + 2t$

Ansatz

$$x(t) = At^2 + Bt + C,$$

Upon substitution into the ode, we obtain

$$2A + 3(2At + B) + 2(At^2 + Bt + C) = t^2 + 2t,$$

from which we obtain

$$A = \frac{1}{2} \qquad B = -\frac{1}{2} \qquad C = \frac{1}{4}.$$

Therefore the particular solution is

$$x_p = \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}.$$

(g) $\ddot{x} - x = \cosh t$

Since the corresponding homgeneous equation has solution

$$x(t) = Ae^t + Be^{-t},$$

it would fail if we try the ansatz as above form. Thereform we should ansatz

$$x(t) = Ate^t + Bte^{-t},$$

with second derivative given by

$$\ddot{x} = 2Ae^t + Ate^t - 2Be^{-t} + Bte^{-t}.$$

Substitution into the differential equation yields

$$2Ae^t - 2B^{-t} = \cosh t = \frac{e^t + e^{-t}}{2},$$

which results in A = 1/4, B = -1/4. Therefore the particular solution is

$$x_p = \frac{1}{4}te^t - \frac{1}{4}te^{-t} \qquad or \qquad x_p = \frac{1}{2}t\sinh t$$

4. **(Practice)** Solve the initial value problem:

$\ddot{x} + 3\dot{x} + 2x = e^{-2t}, \quad x(0) = 0,\ \dot{x}(0) = 0.$

The solution of the homogeneous equation is

$$x_h(t) = Ae^{-2t} + Be^{-t}.$$

and from 3.(b) we have the particular solution is

$$x_p(t) = -te^{-2t}.$$

The general solution is

$$\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= Ae^{-2t} + Be^{-t} - te^{-2t}. \end{aligned}$$

Taking the derivative,

$$\dot{x} = -2Ae^{-2t} - Be^{-t} - e^{-2t} + 2te^{-2t}.$$

Applying the initial conditions,

$$A + B = 0 \qquad -2A - B - 1 = 0,$$

or

$$A = -1 \qquad B = 1.$$

Therefore the solution is

$$x(t) = -e^{-2t} + e^{-t} - te^{-2t}.$$