Solutions to "Week 03 Worksheet"*

Second-Order Homogeneous ODE's

by Xiaoyu Wei

Web: wei.wxyzg.com

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1. **(Demonstration)** Model the case of two real roots by solving for $x = x(t)$:

   $\ddot{x} + 4\,\dot{x} + 3\,x = 0; \quad x(0) = 1,\ \dot{x}(0) = 1.$

   Solution: The characteristic equation found by looking for solutions of the form $x = e^{rt}$ is

$$\begin{aligned} r^2 + 4\,r + 3 &= 0 \\ (r+1)(r+3) &= 0. \end{aligned}$$

   Therefore, with two roots $r_1 = -1$, $r_2 = -3$, the general solution to the equation is

$$x(t) = c_1\,e^{-t} + c_2\,e^{-3t}, \quad c_1, c_2 \in \mathbb{C}.$$

   Fixing constants by initial conditions,

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 3c_2 = 1 \end{cases} \Longrightarrow \quad c_1 = 2, c_2 = -1.$$

   Therefore, the solution is

$$x(t) = 2e^{-t} - e^{-3t}.$$

2. **(Demonstration)** Model the case of complex conjugate roots by solving for $x = x(t)$:

   $\ddot{x} - 2\,\dot{x} + 5\,x = 0; \quad x(0) = 1,\ \dot{x}(0) = 1.$

   Solution: The characteristic equation is

$$\begin{aligned} r^2 - 2r + 5 &= 0 \\ r &= \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i. \end{aligned}$$

   Therefore, the general solution is

$$x(t) = c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t}, \quad c_1, c_2 \in \mathbb{C}.$$

   Plugging into initial conditions

$$\begin{cases} c_1 + c_2 = 1 \\ (1+2i)c_1 + (1-2i)c_2 = 1 \end{cases} \Longrightarrow \quad c_1 = c_2 = \frac{1}{2}.$$

*. This document has been written using the GNU $\mathrm{T_EX_{MACS}}$ text editor (see www.texmacs.org).

This yields the solution

$$\begin{aligned} x(t) &= \frac{1}{2}e^{(1+2i)t} + \frac{1}{2}e^{(1-2i)t} \\ &= \frac{e^t}{2}(e^{2it} + e^{-2it}) \\ &= e^t \cos(2t). \end{aligned}$$

3. **(Demonstration)** Model the case of degenerate real roots by solving for $x = x(t)$:

   $\ddot{x} + 4\,\dot{x} + 4\,x = 0; \quad x(0) = 1,\ \dot{x}(0) = 1$

   Solution: Characteristic equation

$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0. \end{aligned}$$

   Then the general solution follows the ansatz of order reduction

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}, \quad c_1, c_2 \in \mathbb{C}.$$

   From initial conditions,

$$\begin{cases} c_1 = 1 \\ -2c_1 + c_2 = 1 \end{cases} \implies c_1 = 1, c_2 = 3.$$

   Therefore, the solution is

$$x(t) = e^{-2t}(3t+1).$$

   Remark 1. What if I forgot the ansatz? Pertubate the equation a little to be

$$\ddot{x} + 4\,\dot{x} + (4+\varepsilon)\,x = 0,$$

   where $\varepsilon \in \mathbb{C}, 0 < |\varepsilon| \ll 1$. The characteristic equation now has two distinct roots

$$r = -2 \pm \delta,$$

   where $\delta = \sqrt{-\varepsilon}$, thus $0 < |\delta| \ll 1$ too. Now, the general solution to pertubated equation is good old

$$x(t) = c_1 e^{-(2+\delta)t} + c_2 e^{-(2-\delta)t}, \quad c_1, c_2 \in \mathbb{C}.$$

   Suppose the initial conditions are the same, we have

$$\begin{cases} c_1 + c_2 = 1 \\ (-2-\delta)c_1 + (-2+\delta)c_2 = 1 \end{cases} \implies c_1 = \frac{\delta - 3}{2\delta}, c_2 = \frac{\delta + 3}{2\delta}.$$

Then

$$\begin{aligned} x_\delta(t) &= \frac{\delta-3}{2\delta}e^{-(2+\delta)t} + \frac{\delta+3}{2\delta}e^{-(2-\delta)t} \\ &= \frac{1}{2}\big[e^{-(2+\delta)t} + e^{-(2-\delta)t}\big] + \frac{3}{2}e^{-2t}\,\frac{-e^{-\delta t}+e^{\delta t}}{\delta}. \end{aligned}$$

Let $\varepsilon \longrightarrow 0, i.e., \delta \longrightarrow 0$, using L'Hôpital's rule to evaluate the second term, we have

$$\begin{aligned} \lim_{\delta\longrightarrow 0} x_\delta(t) &= \lim_{\delta\longrightarrow 0}\left[e^{-2t} + \frac{3}{2}e^{-2t}\,\frac{t\,e^{-\delta t}+t\,e^{\delta t}}{1}\right] \\ &= e^{-2t} + 3te^{-2t}. \end{aligned}$$

The limiting process can be justified by continuity arguments.

Alternatively, we can figure out what the general solution becomes when $\varepsilon \longrightarrow 0$, by letting $\delta \longrightarrow 0$ in the expression above

$$\begin{aligned} x_\delta(t) &= c_1e^{-(2+\delta)t} + c_2e^{-(2-\delta)t} \\ &= e^{-2t}(c_1e^{-\delta t} + c_2e^{\delta t}) \\ &= e^{-2t}\left[c_1\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\delta^n t^n + c_2\sum_{n=0}^{\infty}\frac{1}{n!}\delta^n t^n\right] \\ &= e^{-2t}\left[(c_1+c_2)\sum_{k=0}^{\infty}\frac{1}{(2k)!}\delta^{2k}t^{2k} + (-c_1+c_2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\delta^{2k+1}t^{2k+1}\right] \\ &= e^{-2t}[(c_1+c_2)\cosh(\delta t) + (-c_1+c_2)\sinh(\delta t)]. \end{aligned}$$

Since $\lim_{x\longrightarrow 0}\cosh(x) = 1$, $\lim_{x\longrightarrow 0}\frac{\sinh(x)}{x} = 1$, we have

$$\lim_{\delta\longrightarrow 0} x_\delta(t) = e^{-2t}(A + Bt),$$

where $A = (c_1+c_2), B = (-c_1+c_2)\,\delta$ can be treated as constants at the limit.

4. **(Practice)** Solve for $x = x(t)$:

   a) $\ddot{x} + 3\,\dot{x} + 2\,x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

   b) $\ddot{x} - 3\,\dot{x} + 2\,x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

   c) $\ddot{x} - 2\,\dot{x} + 2\,x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

   d) $\ddot{x} + 2\,\dot{x} + 2\,x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

   e) $\ddot{x} + 2\,\dot{x} + x = 0, \quad x(0) = 1,\ \dot{x}(0) = 0$

   f) $\ddot{x} - 2\,\dot{x} + x = 0, \quad x(0) = 0,\ \dot{x}(0) = 1$

Solution: Check your answers with the following:

```
Sage] var('t'); x = function('x')(t)
Sage] ### For a) ###
Sage] de1 = diff(x,t,2)+3*diff(x,t)+2*x
Sage] simplify( desolve(de1, x, ics=[0,0,1]) )
```
$e^{(-t)} - e^{(-2\,t)}$

```
Sage] ### For b) ###
Sage] de2 = diff(x,t,2)-3*diff(x,t)+2*x
Sage] simplify( desolve(de2, x, ics=[0,1,0]) )
```
$-e^{(2\,t)} + 2\,e^{t}$

```
Sage] ### For c) ###
Sage] de3 = diff(x,t,2)-2*diff(x,t)+2*x
Sage] simplify( desolve(de3, x, ics=[0,1,0]) )
```
$(\cos(t) - \sin(t))\,e^{t}$

```
Sage] ### For d) ###
Sage] de4 = diff(x,t,2)+2*diff(x,t)+2*x
Sage] simplify( desolve(de4, x, ics=[0,0,1]) )
```
$e^{(-t)}\sin(t)$

```
Sage] ### For e) ###
Sage] de5 = diff(x,t,2)+2*diff(x,t)+x
Sage] simplify( desolve(de5, x, ics=[0,1,0]) )
```
$(t+1)\,e^{(-t)}$

```
Sage] ### For f) ###
Sage] de6 = diff(x,t,2)-2*diff(x,t)+x
Sage] simplify( desolve(de6, x, ics=[0,0,1]) )
```
$t\,e^{t}$

Table 1. Solutions to Problem 4.

5. **(Practice)** Solve the initial value problem $\ddot{x} - \dot{x} - 2\,x = 0, \quad x(0) = \alpha, \dot{x}(0) = 2$ and determine the constant α so that the solution tends to zero as $t \to \infty$.

   Solution: First solve for the solution using techniques demonstrated above,

```
Sage] var('t','alpha'); x = function('x')(t)
Sage] de = diff(x,t,2) - diff(x,t) - 2*x
Sage] simplify( desolve(de, x, ics=[0,alpha,2]) )
```

$$\frac{1}{3}\,(\alpha+2)\,e^{(2\,t)} + \frac{2}{3}\,(\alpha-1)\,e^{(-t)}$$

   To have $x \longrightarrow 0$ as $t \longrightarrow \infty$, the term that blows up mush vanish, i.e.

$$\alpha + 2 \;=\; 0.$$

   Therefore, $\alpha = -2$.

6. **(Practice)** Solve the initial value problem $4\,\ddot{x} - x = 0, \quad x(0) = 2, \dot{x}(0) = \beta$ and determine the constant β so that the solution tends to zero as $t \to \infty$.

   Solution: The equation is still of second order, homogeneous, with constant coefficients.

```
Sage] var('t','beta'); x = function('x')(t)
Sage] de = 4*diff(x,t,2) - x
Sage] simplify( desolve(de, x, ics=[0,2,beta]) )
```

$(\beta+1)\, e^{\left(\frac{1}{2}t\right)} - (\beta-1)\, e^{\left(-\frac{1}{2}t\right)}$

To have $x \longrightarrow 0$ as $t \longrightarrow \infty$, the term that blows up mush vanish, i.e.

$$\beta + 1 \;=\; 0.$$

Therefore, $\beta = -1$.

7. **(Practice)** Find a differential equation whose general solution is given by $x = c_1\, e^{2t} + c_2\, e^{-3t}$, where c_1, c_2 are constants.

Solution: Generally speaking, this is the inverse problem of solving an ODE, which can be thought of as a process of eliminating free constants.

In the first place, since there are two free constants, the equation is probably of second order. We have the following ingradients

$$\begin{aligned} x(t) &= c_1\, e^{2t} + c_2\, e^{-3t}, \\ x'(t) &= 2c_1\, e^{2t} - 3c_2\, e^{-3t}, \\ x''(t) &= 4c_1\, e^{2t} + 9c_2\, e^{-3t}. \end{aligned}$$

For a second order ODE $x'' + A\,x' + B\,x + C = 0$ to be applied, we have

$$\begin{cases} 4c_1 + 2A\, c_1 + B\, c_1 + C \;=\; 0 \\ 9c_2 - 3\mathrm{A}c_2 + B\, c_2 + C \;=\; 0 \end{cases}$$

holds for $\forall c_1, c_2$, thus

$$\begin{cases} C &= 0 \\ 4 + 2A + B &= 0 \\ 9 - 3A + B &= 0 \end{cases} \quad \Longrightarrow \quad A = 1, B = -6, C = 0.$$

Thus,

$$x'' + x' - 6x \;=\; 0.$$

Alternatively, realizing that $x(t)$ is general solution of a homogeneous second order ODE with constant coefficients and knowing both roots, we can directly write the characteristic equation as

$$\begin{aligned} (r-2)(r+3) &= 0 \\ r^2 + r - 6 &= 0. \end{aligned}$$

Therefore the ODE can be

$$x'' + x' - 6x \;=\; 0.$$