Solutions to "Week 02 Worksheet"*

Linear First-Order ODE's and Applications

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1. **(Demonstration)** Model the following first-order linear ode for $y = y(x)$:

   $x\,y' + y = e^x, \qquad y(1) = 0.$

   Solution: This problem is associated with an exact first-order ODE. To see that, rewirte the eqaution as

   $$x\,dy + (y - e^x)dx = 0. \tag{1}$$

   Since $\frac{\partial}{\partial x}(x) = \frac{\partial}{\partial y}(y - e^x) = 1$, the equation (1) is said to be exact, i.e., a scalar potential $H(x, y)$ can be defined, s.t.

   $$H(x, y) = c$$

   is the general solution to the ODE, where c is a constant. In particular,

   $$H(x, y) = \int x\,dy + (y - e^x)dx = x\,y - e^x + C,$$

   where C the constant of integration, which can be set as 0 without loss of generality. Then the solution is

   $$x\,y - e^x = c,$$

   with c determined by initial conditions $c = H(1, 0) = -e$.

   Therefore, the solution to the I.V.P. is

   $$y = \frac{e^x - e}{x}.$$

2. **(Practice)** Solve linear odes for $y = y(x)$.

   a) $x^2\,y' = 1 - 2\,x\,y, \quad y(1) = 2$

   b) $x^4\,y' + 4\,x^3\,y = e^{-x}, \quad y(1) = -1/e$

   c) $y' + 2\,x\,y = x, \quad y(0) = 1/2$

   d) $(1 + x^2)\,y' + 2\,x\,y = 2\,x, \quad y(0) = 0$

   e) $y' + \lambda\,y = a + b\,e^{-\lambda x}, \quad y(0) = 0 \quad (\lambda > 0)$

*. This document has been written using the GNU $\TeX_{\text{MACS}}$ text editor (see www.texmacs.org).

Solution: a), b), d) are exact, c) is separable, and e) can be made exact by multiplying with $e^{\lambda x}$. Solve them as demonstrated above (for c, refer to the previous worksheet) and check your answers:

```
Sage] y = function('y')(x)
Sage] ### For a) ###
Sage] de1 = x^2*diff(y,x) - 1 + 2*x*y
Sage] simplify( desolve(de1, y, ics=[1,2]) )
```

$$\frac{x+1}{x^2}$$

```
Sage] ### For b) ###
Sage] de2 = x^4*diff(y,x) + 4*x^3*y - exp(-x)
Sage] simplify( desolve(de2, y, ics=[1,-1/e]) )
```

$$-\frac{e^{(-x)}}{x^4}$$

```
Sage] ### For c) ###
Sage] de3 = diff(y,x) + 2*x*y - x
Sage] simplify( desolve(de3, y, ics=[0,1/2]) )
```

$$\frac{1}{2}$$

```
Sage] ### For d) ###
Sage] de4 = (1+x^2)*diff(y,x) + 2*x*y - 2*x
Sage] simplify( desolve(de4, y, ics=[0,0]) )
```

$$\frac{x^2}{x^2+1}$$

```
Sage] ### For e) ###
Sage] l = var('l', latex_name=r'\lambda')
Sage] var('a','b')
Sage] assume(l>0)
Sage] de5 = diff(y,x) + l*y - a - b*exp(-l*x)
Sage] simplify( desolve(de5, y, ics=[0,0], ivar=x) )
```

$$\frac{\left(b\,\lambda\,x+a\,e^{(\lambda x)}-a\right)e^{(-\lambda x)}}{\lambda}$$

Remark 1. Checklist for solving **first order** ODE's $y' = F(x,y)$:

I. Is it separable? $F(x,y) \overset{?}{=} X(x)Y(y)$ (Week 01's demo)

II. Is it exact in obvious ways? (Week 02's demo)

III. If not separable nor exact, try to find an integrating factor:

   i. Is it linear? If it is, i.e., of the form $y' + p(x)\,y = q(x)$, then an integrating factor is
   $$M(x) = e^{\int p(x)dx}.$$
   (All first order linear ODE's can be solved in this way).

   ii. Otherwise, try to solve the equations for integrating factors (sometimes it is easier with tricks like change of variables, but sometimes it is as hard as solving the orginal ODE).

3. **(Practice)** Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

   a) Find the T required for the original sum to double in value as a function of r.

b) Determine T if $r = 7\%$.

c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

Solution: Denote $y(x)$ to be the total value of the initial investment and the interest, as a function of time x, such that $y(0) = S_0$. Then since the rate of return is compounded continuously, the model of capital growth can be written as

$$\begin{cases} y' &= r\,y \\ y(0) &= S_0 \end{cases}.$$

Solving this system yields $y(x) = S_0\, e^{rx}$.

```
Sage] y = function('y')(x)
Sage] var('r','S0')
Sage] de = diff(y,x) - r*y
Sage] simplify( desolve(de, y, ics=[0,S0], ivar=x) )
```

$S_0\, e^{(rx)}$

```
Sage] numerical_approx( log(2)/0.07 )
```

9.90210257942779

```
Sage] numerical_approx( log(2)/8 )
```

0.0866433975699932

```
Sage]
```

Now, we have enough power to answer the questions:

a) T should be the solution to the equation $y(T) = 2\,S_0$, i.e.

$$S_0\, e^{rT} = 2S_0,$$

yielding $T = \frac{\ln 2}{r}$.

b) Plugging in, we have $T = \frac{\ln 2}{0.07} \approx 9.9$.

c) r should be the solution to the equation $y(8) = 2\,S_0$, i.e.

$$S_0\, e^{8r} = 2S_0,$$

yielding $r = \frac{\ln 2}{8} \approx 8.7\%$.

4. **(Practice)** A home buyer can afford to spend no more than \$ 800 / month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

   1. Determine the maximum amount that this buyer can afford to borrow.
   2. Determine the total interest paid during the term of the mortgage.

   Solution: Denote $y(x)$ the total amount that this buyer owes as a function of time x, such that $y(0) = M$ being the amount being borrowed at the time of buying the house. Then since the rate of interest is compounded continuously and that the payments are also continuous, the model of debt paying-off can be written as

$$\begin{cases} y' &= 0.09\,y - k \\ y(0) &= M \end{cases},$$

where k is the annual amount of payment, and $k \leqslant 800 \times 12$. Solving this gives

$$y(x) = M\left(1 - \frac{k/M}{0.09}\right)e^{0.09x} + \frac{k}{0.09}.$$

```
Sage] y = function('y')(x)
Sage] var('M','k')
Sage] de = diff(y,x) - 0.09*y + k
Sage] simplify( desolve(de, y, ics=[0,M], ivar=x) )
```

$$M e^{\left(\frac{9}{100}x\right)} - \frac{100}{9}\,k\,e^{\left(\frac{9}{100}x\right)} + \frac{100}{9}\,k$$

```
Sage] numerical_approx( 800*12/0.09 * (exp(1.8)-1)/exp(1.8) )
```

89034.7852563641

```
Sage] numerical_approx( 20 - 1/0.09 * (exp(1.8)-1)/exp(1.8)  )
```

10.7255432024621

```
Sage] numerical_approx( 800*12 * (20 - 1/0.09*(exp(1.8)-1)/exp(1.8)) )
```

102965.214743636

a) The fact that the mortgage being paid off in 20 years indicates $y(20) = 0$, i.e.

$$M\left(1 - \frac{k/M}{0.09}\right)e^{1.8} + \frac{k}{0.09} = 0.$$

It is equivalent to

$$M = \frac{k(e^{1.8} - 1)}{0.09\,e^{1.8}}.$$

Clearly, the maximum of M is attained when the buyer pays as much as possible each month, i.e., when $k = 800 \times 12$.

$$\max_k M \approx 89034.8.$$

b) The total interest paid I equals to the total payment substracts the initial debt,

$$\begin{aligned} I &= 20\,k - M \\ &= k\left(20 - \frac{e^{1.8} - 1}{0.09\,e^{1.8}}\right) \\ &\approx 10.73\,k. \end{aligned}$$

And when $k = 800 \times 12$, this amounts to \$102965.2.

5. **(Practice)** Find the escape velocity for a body projected upward with an initial velocity v_0 from a point $x_0 = \xi\,R$ above the center of the earth, where R is the radius of the earth and ξ is a constant greater than unity. Neglect air resistance. Find the initial altitude from which the body must be launched in order to reduce the escape velocity to 85% of its value at the earth's surface.
Solution: Since the object is projected upward, the motion is actually 1D. From Newton's law of universal gravitation, the force applied to the body by the earth is

$$F = -G\frac{m\,M}{r^2},$$

where G is the gravitational constant, m is the mass of the object, M is the mass of Earth, r is the distance bewteen the object and the center of the earth.
On the other hand, considering the motion as a function of time t, from Newton's second law of motion,

$$F = m\,r'',$$

where $r'' = \frac{d^2 r}{dt^2}$, we have

$$r'' = -G\frac{M}{r^2}.$$

The initial conditions are initial position and initial velocity

$$r(0) = r_0, \quad r'(0) = v_0.$$

This ODE is second-order and not easy to solve, but we can make a change of variables to simplify it. Noticing that time t is not explicitly included in the equation, we may use r as new independent variable, and $v = \frac{dr}{dt}$ as new dependent variable, then

$$r'' = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v'\,v.$$

Note the difference in meaning of "prime" here. The ODE becomes first-order and separable,

$$v\,v' = -\frac{G\,M}{r^2}.$$

In the mean while, the initial condition becomes $v(r_0) = v_0$.

```
Sage] v = function('v')(r)
Sage] var('G','M','r','r0','v0')
```

(G, M, r, r0, v0)

```
Sage] de = v*diff(v,r) + G*M/r^2
Sage] simplify( desolve(de, v, ics=[r0,v0], ivar=r) )
```

$$-\frac{v(r)^2}{2\,G\,M} = -\frac{r\,r_0\,v_0^2 - 2\,G\,M\,(r - r_0)}{2\,G\,M\,r\,r_0}$$

The solution is thus

$$v^2 = v_0^2 - 2\,G\,M\,\frac{r - r_0}{r\,r_0}. \tag{2}$$

Initially, the object is projected with $v_0 > 0$. To escape, the velocity must never decrease to zero, that is,

$$v_0^2 - 2\,G\,M\,\frac{r - r_0}{r\,r_0} > 0, \quad \forall r \geqslant r_0,$$

so

$$v_0^2 \geqslant 2\,G\,M\,\frac{1}{r_0}.$$

That is to say, the escape velocity is $v_e = \sqrt{\frac{2\,GM}{r_0}}$. With $r_0 = \xi\,R$, we have the equation

$$\sqrt{\frac{2\,GM}{\xi\,R}} = 0.85\,\sqrt{\frac{2\,GM}{R}},$$

yielding the value of ξ to be 1.384.

```
Sage] numerical_approx(1/0.85^2)
```

1.38408304498270

```
Sage] numerical_approx((1/0.85^2 - 1)*6371)
```

2446.99307958478

So the initial altitude should be

$$(\xi - 1)\,R \approx 0.384 \times 6371 \approx 2447\,(\mathrm{km}).$$

Remark 2. The solution (2) is nothing but energy conservation

$$\frac{1}{2}mv_0^2 - G\frac{Mm}{r_0} = \frac{1}{2}mv^2 - G\frac{Mm}{r}.$$

Solution is much more intuitive to obtain given this insight.

6. **(Practice)** A tank initially contains an amount S (liters) of pure water. A mixture containing a concentration γ (grams/liter) of salt enters the tank at a rate r (liters/minute), and the well-stirred mixture leaves the tank at the same rate.

   a) Determine a differential equation for the amount of salt $M(t)$ (grams) in the tank at any time t by writing an equation for $M(t+\Delta t)$.

   b) Solve this differential equation using an integrating factor.

   c) Find the limiting amount of salt in the tank as $t \to \infty$, and show that this corresponds to the solution obtained by setting $dM/dt = 0$.

   Solution:

   a) Since inflow rate equals to outflow rate, the amount of liquid in the container is constant. Assuming the concentration of salt is constant within infinitesimal Δt, we have

   $$M(t+\Delta t) - M(t) = \gamma r \,\Delta t - \frac{M(t)}{S} r \,\Delta t,$$

   that is,

   $$\frac{M(t+\Delta t) - M(t)}{\Delta t} = \gamma r - \frac{M(t)}{S} r.$$

   Take the limit $\Delta t \longrightarrow 0$,

   $$\frac{M'(t)}{r} = \gamma - \frac{M(t)}{S}.$$

   The ODE above is subject to the initial condition $M(0) = 0$, since initially there is only pure water in the tank.

   b) Using the integrating factor

   $$f(t) = e^{\int \frac{r}{S} dt} = e^{\frac{rt}{S}},$$

   the ODE becomes exact

   $$e^{\frac{rt}{S}} \frac{M'}{r} = \gamma e^{\frac{rt}{S}} - \frac{e^{\frac{rt}{S}}}{S} M.$$

   This is equivalent to

   $$\left(e^{\frac{rt}{S}} M\right)' = \gamma r e^{\frac{rt}{S}},$$

   and the solution is

   $$M(t) = e^{-\frac{rt}{S}} \int_0^t \gamma r e^{\frac{rz}{S}} dz = S\gamma\left(1 - e^{-\frac{t}{S}}\right).$$

```
Sage] M = function('M')(t)
Sage] var('S','gamma','r','t')
    (S, gamma, r, t)
Sage] de = diff(M,t)/r + M/S == gamma
Sage] simplify( desolve(de, M, ics=[0,0], ivar=t) )
```

   $$\left(S\gamma e^{\left(\frac{rt}{S}\right)} - S\gamma\right) e^{\left(-\frac{rt}{S}\right)}$$

   c) Setting $t \longrightarrow \infty$, $M(t) \longrightarrow S\gamma$. On the other hand, in the ODE, setting $M' = 0$ also yields

   $$0 = \gamma - \frac{M}{S} \quad \Longrightarrow \quad M = S\gamma.$$

7. **(Practice)** A spherical raindrop evaporates at a rate proportional to its surface area. By finding and solving a differential equation for the radius of the drop, show that the radius decreases linearly with time. Use the following facts to help you solve this problem:
   (i) The volume of a raindrop is $\frac{4}{3}\pi r^3$.

(ii) The surface area of a raindrop is $4\,\pi\,r^2$.
(iii) The time-derivative of the volume is proportional to the surface area.
Solution: Denote $r(t)$ the radius of the drop, and assmue the rate of evaporation is μ times of its surface area, s.t.

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \mu\,(4\pi r^2).$$

That is,

$$r^2\,r' = \mu r^2 \quad \Longrightarrow \quad r' = \mu.$$

Therefore, the radius decreases linearly with time in this model.