Solutions to "Week 01 Worksheet"*

Complex Numbers and Separable First-Order ODE's

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1. **(Demonstration)** Complex number problems of various sorts.

2. **(Demonstration)** Model the following first-order separable ode for $y = y(x)$:

$$(1+y)\, y' = x, \qquad y(0) = 0$$

Solution: (This is known as D'Alembert's equation) First let's try to separate x and y, by writing the equation into

$$(1+y)\frac{dy}{dx} = x,$$

which is equivalent to

$$(1+y)\, dy = x\, dx.$$

Now, the left hand side explicitly depends on y only, while the right hand side explicitly depends on x only. To get the solution of the orginal ODE, we intend to calculate the value of $y(x)$ as a function of x. To do that, we can integrate both sides from the initial point $x = 0, y = 0$ to an arbitrary end point $x = x_\star, y = y_\star$ to find out the dependence between $x_\star$ and $y_\star$,

$$\int_{y=0}^{y=y_\star} (1+y)\, dy = \int_{x=0}^{x=x_\star} x\, dx.$$

Realizing that the separation of x and y allows us to calculate the integrals explicitly, we have

$$\frac{1}{2}(1+y_\star)^2 - \frac{1}{2} = \frac{1}{2}x_\star^2,$$

that is,

$$(1+y_\star)^2 - 1 = x_\star^2.$$

Reinterpretating the results, we can say that the solution to the original initial value problem is a hyperbola (as in Figure 1) defined by the quadratic equation

$$(y+1)^2 - x^2 = 1.$$

*. This document has been written using the GNU TeXmacs text editor (see `www.texmacs.org`).

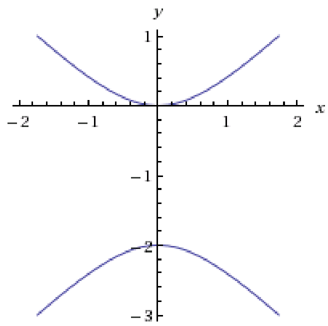

Figure 1. Solution curve.

Remark 1. If you take take the solution further to be

$$y = -1 \pm \sqrt{x^2+1},$$

then it is suggested to re-check the initial condition on both branches. As you can see in Figure 1, only the upper branch satisfies the initial condition, so the solution should be

$$y = -1 + \sqrt{x^2+1}.$$

(In this case the two branches are connected smoothly at infinity, you may also argue in this way and keep both branches).

3. **(Practice)** Write as a complex number $z = x + i\,y$, where x and y are real.

   a) $\dfrac{1+3\,i}{3-2\,i}$

   b) $\dfrac{1}{1+i} + \dfrac{1}{1-i}$

   c) $\dfrac{-1-2\,i}{-4+3\,i}$

   d) $-(7-i)(-4-2\,i)(2-i)$

Solution: The idea to simplify is,

i. Addition/substraction:

$$(x_1 + i\,y_1) \pm (x_2 + i\,y_2) = (x_1 \pm x_2) + i\,(y_1 \pm y_2).$$

ii. Multiplication:

$$(x_1 + i\,y_1) \star (x_2 + i\,y_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

iii. Division: (first make denominator real by multiplying with its conjuate)

$$\frac{(x_1 + i\,y_1)}{(x_2 + i\,y_2)} = \frac{(x_1 + i\,y_1)(x_2 - i\,y_2)}{(x_2 + i\,y_2)(x_2 - i\,y_2)} = \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + x_2y_1)}{(x_2^2 + y_2^2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2}.$$

Apply those rules, and verify your answer with below (the programs are excutable in Sage, an open-source computer algebra system, and you are encouraged to play with them at home):

```
Sage] simplify( (1+3*I)/(3-2*I) ) # See answer for a) below:
```
$\frac{11}{13}i - \frac{3}{13}$
```
Sage] simplify( 1/(1+I) + 1/(1-I) ) # See answer for b) below:
```
1
```
Sage] simplify( (-1-2*I)/(-4+3*I) ) # See answer for c) below:
```
$\frac{11}{25}i - \frac{2}{25}$
```
Sage] simplify( -(7-I)*(-4-2*I)*(2-I) ) # See answer for d) below:
```
$-10\,i + 70$

Table 1. Solutions to Problem 3.

4. **(Practice)** Convert to polar form $z = r\exp(i\,\theta)$.

   a) $1 + \sqrt{3}\,i$

   b) $(\sqrt{2} + \sqrt{2}\,i)^7$

Solution: By Euler's formula, for any real number θ, we have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Therefore, for $z = x + i\,y$, we have $z = |z|\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) = |z|(\cos\theta + i\sin\theta)$, where $\theta = \arccos\frac{x}{\sqrt{x^2+y^2}}$. Then $z = |z|\,e^{i\theta}$.

a). $1 + \sqrt{3}\,i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\,e^{i\frac{\pi}{3}}$.

b). $(\sqrt{2} + \sqrt{2}\,i)^7 = \left[2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right]^7 = 2^7\left(e^{i\frac{\pi}{4}}\right)^7 = 128\,e^{i\frac{7\pi}{4}} = 128\,e^{-i\frac{\pi}{4}}$.

```
Sage] a = 1+sqrt(3)*I
Sage] simplify( abs(a) )
```
2
```
Sage] simplify( arg(a) )
```
$\frac{1}{3}\pi$
```
Sage] b = (sqrt(2)+sqrt(2)*I)^7
Sage] simplify( abs(b) )
```
128
```
Sage] simplify( arg(b) )
```
$-\frac{1}{4}\pi$

Table 2. Solutions to Problem 4.

5. **(Practice)** Solve for x and y, where x and y are real.

   $2\,y + i\,x = 4 + x - i$

   Solution: By taking the real and complex parts of the equation, we have

$$\begin{cases} 2y &= 4+x \\ x &= -1 \end{cases}.$$

   Therefore, $x = -1$, $y = \frac{3}{2}$.

6. **(Practice)** With x real, find the real and the imaginary parts

   $\exp\left((5+12\,i)\,x\right)$

   Solution: Using Euler's formula,

$$\begin{aligned} \exp\left((5+12\,i)\,x\right) &= e^{5x+12ix} \\ &= e^{5x}\,e^{i(12x)} \\ &= e^{5x}[\cos(12x) + i\sin(12x)]. \end{aligned}$$

   Therefore, the real part is

$$\mathrm{real}\left(e^{(5+12i)x}\right) = e^{5x}\cos(12x),$$

   and the imaginary part is

$$\mathrm{imag}\left(e^{(5+12i)x}\right) = e^{5x}\sin(12x).$$

7. **(Practice)** Solve separable odes for $y = y(x)$.

   a) $y' = \sqrt{x\,y}, \quad y(1) = 0$

   b) $y^2 - x\,y' = 0, \quad y(1) = 1$

   c) $e^{x-y}\,y' + e^{y-x} = 0, \quad y(0) = 0$

   d) $y' + (\sin x)\,y = 0, \quad y\,(\pi/2) = 1$

   e) $y' = y\,(1-y), \quad y(0) = y_0 \quad (y_0 > 0)$

   Solution: All the problems here can be solved using the same technique as in Problem 2. If you get into troubles with integration part, you may refer to a list of integrals such as the one on Wikipedia (https://en.wikipedia.org/wiki/Lists_of_integrals).

   Verify your answer with the following:

```
Sage] y = function('y')(x)
Sage] ### For a) ###
Sage] de1 = diff(y,x) - sqrt(x)*sqrt(y)
Sage] simplify( desolve(de1, y, ics=[1,0]) )
```

$$2\sqrt{y(x)} = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}$$

```
Sage] ### For b) ###
Sage] de2 = y^2 - x*diff(y,x)
Sage] simplify( desolve(de2, y, ics=[1,1]) )
```

$$-\frac{1}{y(x)} = \log(x) - 1$$

```
Sage] ### For c) ###
Sage] de3 = diff(y,x) + exp(2*(y-x))
Sage] simplify( desolve(de3, y, ics=[0,0]) )
```

$$-\frac{1}{2}\left(e^{(2x)} + e^{(2y(x))}\right)e^{(-2x-2y(x))} = (-1)$$

```
Sage] ### For d) ###
Sage] de4 = diff(y,x) + sin(x)*y
Sage] simplify( desolve(de4, y, ics=[pi/2,1]) )
```

$$e^{\cos(x)}$$

```
Sage] ### For e)
Sage] var('y0')
```

$$y_0$$

```
Sage] de5 = diff(y,x) - y*(1-y)
Sage] simplify( desolve(de5, y, ics=[0,y0]) )
```

$$-\log(y(x)-1) + \log(y(x)) = x - \log(y_0-1) + \log(y_0)$$

Table 3. Solutions to Problem 7.

And here is a more readable version for the solutions above:

a) $y = \left(\frac{x^{3/2}-1}{3}\right)^2, \quad x \geqslant 1.$

b) $y = \frac{1}{1-\log x}, \quad 1 \leqslant x < e.$

c) $e^{-2x} + e^{-2y} = 2, \quad x \geqslant 0.$

d) $y = e^{\cos x}, \quad x \geqslant \frac{\pi}{2}.$

e) $\frac{y}{y-1} = \exp\left[x + \log\left(\frac{y_0}{y_0-1}\right)\right] = \frac{y_0}{y_0-1}e^x, \quad x \geqslant 0.$