which is exactly the same as the Poisson formula (6.3.14), which we found earlier in a completely different way!

EXERCISES

1. Find the one-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be restated as follows.
   (i) It solves $G''(x) = 0$ for $x \neq x_0$ ("harmonic").
   (ii) $G(0) = G(l) = 0$.
   (iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.
2. Verify directly from (3) or (4) that the solution of the half-space problem satisfies the condition at infinity:
$$u(\mathbf{x}) \to 0 \qquad \text{as } |\mathbf{x}| \to \infty.$$
   Assume that $h(x, y)$ is a continuous function that vanishes outside some circle.
3. Show directly from (3) that the boundary condition is satisfied: $u(x_0, y_0, z_0) \to h(x_0, y_0)$ as $z_0 \to 0$. Assume $h(x, y)$ is continuous and bounded. [*Hint:* Change variables $s^2 = [(x - x_0)^2 + (y - y_0)^2]/z_0^2$ and use the fact that $\int_0^\infty s(s^2 + 1)^{-3/2} ds = 1$.]
4. Verify directly from (3) that the solution has derivatives of all orders in $\{z > 0\}$. Assume that $h(x, y)$ is a continuous function that vanishes outside some circle. (*Hint:* See Section A.3 for differentiation under an integral sign.)
5. Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.
6. (a) Find the Green's function for the half-plane $\{(x, y): y > 0\}$.
   (b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.
   (c) Calculate the solution with $u(x, 0) = 1$.
7. (a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.
   (b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.
   (c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x, y)$ is a function of the quotient x/y.
   (d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.
   (e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.
8. (a) Use Exercise 7 to find the harmonic function in the half-plane $\{y > 0\}$ with the boundary data $h(x) = 1$ for $x > 0$, $h(x) = 0$ for $x < 0$.
   (b) Do the same as part (a) for the boundary data $h(x) = 1$ for $x > a$, $h(x) = 0$ for $x < a$. (*Hint:* Translate the preceding answer.)