This identity (2) is valid for any small $\epsilon > 0$. Our representation formula (1) would follow provided that we could show that the right side of (2) tended to $4\pi u(\mathbf{0})$ as $\epsilon \to 0$.

Now, on the little spherical surface $\{r = \epsilon\}$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} = -\frac{1}{\epsilon^2},$$

so that the right side of (2) equals

$$\frac{1}{\epsilon^2}\iint_{r=\epsilon} u\, dS + \frac{1}{\epsilon}\iint_{r=\epsilon} \frac{\partial u}{\partial r}\, dS = 4\pi\overline{u} + 4\pi\epsilon\overline{\frac{\partial u}{\partial r}}, \tag{3}$$

where $\overline{u}$ denotes the average value of $u(\mathbf{x})$ on the sphere $|\mathbf{x}| = r = \epsilon$, and $\overline{\partial u/\partial r}$ denotes the average value of $\partial u/\partial n$ on this sphere. As $\epsilon \to 0$, the expression (3) approaches

$$4\pi u(\mathbf{0}) + 4\pi \cdot 0 \cdot \frac{\partial u}{\partial r}(\mathbf{0}) = 4\pi u(\mathbf{0}) \tag{4}$$

because u is continuous and $\partial u/\partial r$ is bounded. Thus (2) turns into (1), and this completes the proof.

The corresponding formula in *two dimensions* is

$$\boxed{u(\mathbf{x}_0) = \frac{1}{2\pi}\int_{\text{bdy } D}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\log|\mathbf{x}-\mathbf{x}_0|) - \frac{\partial u}{\partial n}\log|\mathbf{x}-\mathbf{x}_0|\right] ds} \tag{5}$$

whenever $\Delta u = 0$ in a plane domain D and $\mathbf{x}_0$ is a point within D. The right side is a line integral over the boundary curve with respect to arc length. Log denotes the natural logarithm and *ds* the arc length on the bounding curve.

EXERCISES

1. Derive the representation formula for harmonic functions (7.2.5) in two dimensions.
2. Let $\phi(\mathbf{x})$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

   The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.
3. Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and $\mathbf{x}_0$ its center in the representation formula (1).