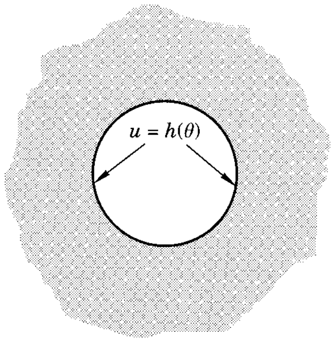

Figure 3

The boundary condition means

$$h(\theta) = \tfrac{1}{2}A_0 + \sum a^{-n}(A_n \cos n\theta + B_n \sin n\theta),$$

so that

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta \, d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta \, d\theta.$$

This is the complete solution but it is one of the rare cases when the series can actually be summed. Comparing it with the interior case, we see that the only difference between the two sets of formulas is that r and a are replaced by r^{-1} and a^{-1}. Therefore, we get Poisson's formula with only this alteration. The result can be written as

$$\boxed{u(r,\theta) = (r^2 - a^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,\frac{d\phi}{2\pi}} \qquad (9)$$

for $r > a$. □

These three examples illustrate the technique of separating variables in polar coordinates. A number of other examples are given in the exercises. What is the most general domain that can be treated by this method?

EXERCISES

1. Solve $u_{xx} + u_{yy} = 0$ in the *exterior* $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.