EXERCISES

1. Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.
   (a) Find the maximum value of u in $\overline{D}$.
   (b) Calculate the value of u at the origin.
2. Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition
$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$
3. Same for the boundary condition $u = \sin^3\theta$. (*Hint:* Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)
4. Show that $P(r, \theta)$ is a harmonic function in D by using polar coordinates. That is, use (6.1.5) on the first expression in (17).

6.4 CIRCLES, WEDGES, AND ANNULI

The technique of separating variables in polar coordinates works for domains whose boundaries are made up of concentric circles and rays. The purpose of this section is to present several examples of this type. In each case we get the expansion as an infinite series. (But summing the series to get a Poisson-type formula is more difficult and works only in special cases.) The geometries we treat here are

A wedge: $\{0 < \theta < \theta_0, 0 < r < a\}$
An annulus: $\{0 < a < r < b\}$
The exterior of a circle: $\{a < r < \infty\}$

We could do Dirichlet, Neumann, or Robin boundary conditions. This leaves us with a lot of possible examples!

Example 1. The Wedge

Let us take the wedge with three sides $\theta = 0$, $\theta = \beta$, and $r = a$ and solve the Laplace equation with the homogeneous Dirichlet condition on the straight sides and the inhomogeneous Neumann condition on the curved side (see Figure 1). That is, using the notation $u = u(r, \theta)$, the BCs are

$$\boxed{u(r, 0) = 0 = u(r, \beta), \qquad \frac{\partial u}{\partial r}(a, \theta) = h(\theta).} \tag{1}$$

The separation-of-variables technique works just as for the circle, namely,

$$\Theta'' + \lambda\Theta = 0, \qquad r^2R'' + rR' - \lambda R = 0.$$