(*Hint:* Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

2. Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$.
3. Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \quad \text{for } y = 0 \text{ and for } y = \pi, \quad u = 0 \quad \text{for } x = 0 \quad \text{and}$$
$$u = \cos^2 y = \tfrac{1}{2}(1 + \cos 2y) \quad \text{for } x = \pi.$$

4. Find the harmonic function in the square $\{0 < x < 1, 0 < y < 1\}$ with the boundary conditions $u(x, 0) = x$, $u(x, 1) = 0$, $u_x(0, y) = 0$, $u_x(1, y) = y^2$.
5. Solve Example 1 in the case $b = 1$, $g(x) = h(x) = k(x) = 0$ but $j(x)$ an arbitrary function.
6. Solve the following Neumann problem in the cube $\{0 < x < 1, 0 < y < 1, 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.
7. (a) Find the harmonic function in the semi-infinite strip $\{0 \le x \le \pi, 0 \le y < \infty\}$ that satisfies the "boundary conditions":

$$u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = h(x), \quad \lim_{y\to\infty} u(x, y) = 0.$$

   (b) What would go awry if we omitted the condition at infinity?

6.3 POISSON'S FORMULA

A much more interesting case is the *Dirichlet problem for a circle*. The rotational invariance of Δ provides a hint that the circle is a natural shape for harmonic functions.

Let's consider the problem

$$\boxed{\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } x^2 + y^2 < a^2 \\ u &= h(\theta) \qquad \text{for } x^2 + y^2 = a^2 \end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

with radius a and any boundary data $h(\theta)$.

Our method, naturally, is to separate variables in *polar* coordinates: $u = R(r)\,\Theta(\theta)$ (see Figure 1). From (6.1.5) we can write

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$
$$= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$