Each quotient X''/X, Y''/Y, and Z''/Z must be a constant. In the familiar way, we find

$$Y(y) = \sin my \quad (m = 1, 2, \ldots)$$

and

$$Z(z) = \sin nz \quad (n = 1, 2, \ldots),$$

so that

$$X'' = (m^2 + n^2)X, \qquad X(0) = 0.$$

Therefore,

$$X(x) = A \sinh(\sqrt{m^2 + n^2}\, x).$$

Summing up, our complete solution is

$$\boxed{u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}\, x) \sin my \sin nz.} \tag{7}$$

Finally, we plug in our inhomogeneous condition at $x = \pi$:

$$g(y, z) = \sum \sum A_{mn} \sinh(\sqrt{m^2 + n^2}\, \pi) \sin my \sin nz.$$

This is a *double* Fourier sine series in the variables y and z! Its theory is similar to that of the single series. In fact, the eigenfunctions $\{\sin my \cdot \sin nz\}$ are mutually orthogonal on the square $\{0 < y < \pi, 0 < z < \pi\}$ (see Exercise 2). Their normalizing constants are

$$\int_0^\pi \int_0^\pi (\sin my \sin nz)^2 \, dy \, dz = \frac{\pi^2}{4}.$$

Therefore,

$$A_{mn} = \frac{4}{\pi^2 \sinh(\sqrt{m^2 + n^2}\, \pi)} \int_0^\pi \int_0^\pi g(y, z) \sin my \sin nz \, dy \, dz. \tag{8}$$

Hence the solutions can be expressed as the doubly infinite series (7) with the coefficients A_{mn}. The complete solution to Example 2 is (7) and (8). With such a series, as with a double integral, one has to be careful about the order of summation, although in most cases any order will give the correct answer. □

EXERCISES

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:

$$\begin{aligned} u_x &= -a \quad \text{on } x = 0 \qquad & u_x &= 0 \quad \text{on } x = a \\ u_y &= b \quad \text{on } y = 0 \qquad & u_y &= 0 \quad \text{on } y = b. \end{aligned}$$