they satisfy the ODE

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r} u_r.$$

So $(r^2 u_r)_r = 0$. It has the solutions $r^2 u_r = c_1$. That is, $u = -c_1 r^{-1} + c_2$. This important harmonic function

$$\frac{\mathbf{1}}{\mathbf{r}} = (x^2 + y^2 + z^2)^{-1/2}$$

is the analog of the special two-dimensional function $\log(x^2 + y^2)^{1/2}$ found before. Strictly speaking, neither function is finite at the origin. In electrostatics the function $u(\mathbf{x}) = r^{-1}$ turns out to be the electrostatic potential when a unit charge is placed at the origin. For further discussion, see Section 12.2.

EXERCISES

1. Show that a function which is a power series in the complex variable $x + iy$ must satisfy the Cauchy–Riemann equations and therefore Laplace's equation.
2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$, where k is a positive constant. (*Hint:* Substitute $u = v/r$.)
3. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u$, where k is a positive constant. (*Hint:* Look up Bessel's differential equation in [MF] or in Section 10.5.)
4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants. (*Hint:* Look for a solution depending only on r.)
5. Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.
6. Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.
7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u(x, y, z)$ vanishing on both the inner and outer boundaries.
8. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u = 0$ on $r = a$ and $\partial u/\partial r = 0$ on $r = b$. Then let $a \to 0$ in your answer and interpret the result.
9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. Its inner boundary is held at 100°C. Its outer boundary satisfies $\partial u/\partial r = -\gamma < 0$, where γ is a constant.
   (a) Find the temperature. (*Hint:* The temperature depends only on the radius.)
   (b) What are the hottest and coldest temperatures?
   (c) Can you choose γ so that the temperature on its outer boundary is 20°C?