EXERCISES

1. $\Sigma_{n=0}^{\infty}(-1)^n x^{2n}$ is a geometric series.
   (a) Does it converge pointwise in the interval $-1 < x < 1$?
   (b) Does it converge uniformly in the interval $-1 < x < 1$?
   (c) Does it converge in the L^2 sense in the interval $-1 < x < 1$?
   (*Hint:* You can compute its partial sums explicitly.)
2. Consider any series of functions on any finite interval. Show that if it converges uniformly, then it also converges in the L^2 sense and in the pointwise sense.
3. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n\left(\frac{1}{2}\right) = 0$, $f_n(x) = \gamma_n$ in the interval $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2})$, let $f_n(x) = -\gamma_n$ in the interval $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ and let $f_n(x) = 0$ elsewhere. Show that:
   (a) $f_n(x) \to 0$ pointwise.
   (b) The convergence is not uniform.
   (c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.
   (d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n = n$.
4. Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\dfrac{1}{4} - \dfrac{1}{n^2}, \dfrac{1}{4} + \dfrac{1}{n^2}\right) & \text{for odd } n \\ 1 \text{ in the interval } \left[\dfrac{3}{4} - \dfrac{1}{n^2}, \dfrac{3}{4} + \dfrac{1}{n^2}\right) & \text{for even } n \\ 0 & \text{for all other } x. \end{cases}$$

   Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.
5. Let $\phi(x) = 0$ for $0 < x < 1$ and $\phi(x) = 1$ for $1 < x < 3$.
   (a) Find the first four nonzero terms of its Fourier cosine series explicitly.
   (b) For each x $(0 \le x \le 3)$, what is the sum of this series?
   (c) Does it converge to $\phi(x)$ in the L^2 sense? Why?
   (d) Put $x = 0$ to find the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots .$$

6. Find the sine series of the function $\cos x$ on the interval $(0, \pi)$. For each x satisfying $-\pi \le x \le \pi$, what is the sum of the series?
7. Let

$$\phi(x) = \begin{cases} -1 - x & \text{for } -1 < x < 0 \\ +1 - x & \text{for } 0 < x < 1. \end{cases}$$