so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l}.}$$

The initial condition requires the cosine expansion (6).

EXERCISES

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.
2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0, t) = 0$, $u(l, \mathrm{t}) = 0$ (Neumann at the left, Dirichlet at the right).
   (a) Show that the eigenfunctions are $\cos[(n + \frac{1}{2})\pi x/l]$.
   (b) Write the series expansion for a solution $u(x,t)$.
3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.
4. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \leq x \leq l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \leq x \leq l$$

$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

   These are called *periodic boundary conditions*.
   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.
   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}.$$

4.3 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$X' - a_0 X = 0 \quad \text{at } x = 0 \tag{1}$$

$$X' + a_l X = 0 \quad \text{at } x = l. \tag{2}$$

The two constants a_0 and a_l should be considered as given.