EXERCISES

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.
   (b) Explain why the note rises when the string is tightened.
2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion
$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$
3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.
4. Consider waves in a resistant medium that satisfy the problem
$$u_{tt} = c^2u_{xx} - ru_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x),$$
   where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.
5. Do the same for $2\pi c/l < r < 4\pi c/l$.
6. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

4.2 THE NEUMANN CONDITION

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0, t) = u_x(l, t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$\boxed{-X'' = \lambda X, \quad X'(0) = X'(l) = 0,} \tag{1}$$

other than the trivial solution $X(x) \equiv 0$.

As before, let's first search for the positive eigenvalues $\lambda = \beta^2 > 0$. As in (4.1.6), $X(x) = C\cos\beta x + D\sin\beta x$, so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$