As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x - y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x - y, t)$ and so is

$$v(x, t) = \int_{-\infty}^{\infty} S(x - y, t) g(y)\, dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x,t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\,x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x, t) = u(\sqrt{a}\,x, at)$. Then $v_t = [\partial(at)/\partial t]u_t = au_t$ and $v_x = [\partial(\sqrt{a}\,x)/\partial x]u_x = \sqrt{a}\,u_x$ and $v_{xx} = \sqrt{a} \cdot \sqrt{a}\,u_{xx} = a\,u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$\boxed{Q(x, 0) = 1 \quad \text{for } x > 0 \quad Q(x, 0) = 0 \quad \text{for } x < 0.} \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x, t) = g(p) \quad \text{where } p = \frac{x}{\sqrt{4kt}} \tag{4}$$