(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1,\ 0 < t < \infty$.
(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x, t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \ge 0$ and $0 \le x \le 1$.
(c) Use the energy method to show that $\int_0^1 u^2\, dx$ is a strictly decreasing function of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.
(b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty,\ 0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f,\ v_t - kv_{xx} = g,\ f \le g$, and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$.
(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi,\ 0 < t < \infty$, and if $v(0, t) \ge 0$, $v(\pi, t) \ge 0$ and $v(x, 0) \ge \sin x$, use part (a) to show that $v(x, t) \ge (1 - e^{-t}) \sin x$.

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\, dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$u_t = ku_{xx} \quad (-\infty < x < \infty,\ 0 < t < \infty) \qquad (1)$$

$$u(x, 0) = \phi(x). \qquad (2)$$