STABILITY

This is the third ingredient of well-posedness (see Section 1.5). It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of problem (3), in case $h = g = f = 0$. Let $u_1(x, 0) = \phi_1(x)$ and $u_2(x, 0) = \phi_2(x)$. Then $w = u_1 - u_2$ is the solution with the initial datum $\phi_1 - \phi_2$. So from (4) we have

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\,dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\,dx. \tag{5}$$

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at $t = 0$), we stay nearby. This is exactly the meaning of stability in the "square integral" sense (see Sections 1.5 and 5.4).

The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle and $w = \phi_1 - \phi_2$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \le \max|\phi_1 - \phi_2|.$$

The "minimum" principle says that

$$u_1(x,t) - u_2(x,t) \ge -\max|\phi_1 - \phi_2|.$$

Therefore,

$$\max_{0\le x\le l} |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|, \tag{6}$$

valid for all $t > 0$. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in the "uniform" sense.

EXERCISES

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1,\ 0 \le t \le T\}$.
2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l,\ 0 \le t < \infty\}$.
   (a) Let $M(T)$ = the maximum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?
   (b) Let $m(T)$ = the minimum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?
3. Consider the diffusion equation $u_t = u_{xx}$ in the interval (0, 1) with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.