uniqueness). It is also unstable. To illustrate the instability further, consider a nonsingular matrix A with one very small eigenvalue. The solution is unique but if b is slightly perturbed, then the error will be greatly magnified in the solution u. Such a matrix, in the context of scientific computation, is called ill-conditioned. The ill-conditioning comes from the instability of the matrix equation with a singular matrix.

As a fourth example, consider Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $D = \{-\infty < x < \infty, 0 < y < \infty\}$. It is *not* a well-posed problem to specify both u and u_y on the boundary of D, for the following reason. It has the solutions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny. \tag{2}$$

Notice that they have boundary data $u_n(x, 0) = 0$ and $\partial u_n/\partial y(x, 0) = e^{-\sqrt{n}} \sin nx$, which tends to zero as $n \to \infty$. But for $y \neq 0$ the solutions $u_n(x, y)$ do not tend to zero as $n \to \infty$. Thus the stability condition (iii) is violated.

EXERCISES

1. Consider the problem

$$\frac{d^2u}{dx^2} + u = 0$$
$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

consisting of an ODE and a pair of boundary conditions. Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution *unique, or not*? Does the answer depend on L?

2. Consider the problem

$$u''(x) + u'(x) = f(x)$$
$$u'(0) = u(0) = \tfrac{1}{2}[u'(l) + u(l)],$$

with $f(x)$ a given function.
   (a) Is the solution *unique*? Explain.
   (b) Does a solution necessarily *exist*, or is there a condition that $f(x)$ must satisfy for existence? Explain.

3. Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?

4. Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D.$$