Example 2.

Find the solution of (4) that satisfies the auxiliary condition $u(0, y) = y^3$. Indeed, putting $x = 0$ in (7), we get $y^3 = f(e^{-0}y)$, so that $f(y) = y^3$. Therefore, $u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3$. □

Example 3.

Solve the PDE

$$\boxed{u_x + 2xy^2u_y = 0.} \tag{8}$$

The characteristic curves satisfy the ODE $dy/dx = 2xy^2/1 = 2xy^2$. To solve the ODE, we separate variables: $dy/y^2 = 2x\,dx$; hence $-1/y = x^2 - C$, so that

$$y = (C - x^2)^{-1}. \tag{9}$$

These curves are the characteristics. Again, $u(x, y)$ is a constant on each such curve. (Check it by writing it out.) So $u(x, y) = f(C)$, where f is an arbitrary function. Therefore, the general solution of (8) is obtained by solving (9) for C. That is,

$$\boxed{u(x, y) = f\left(x^2 + \frac{1}{y}\right).} \tag{10}$$

Again this is easily checked by differentiation, using the chain rule: $u_x = 2x \cdot f'(x^2 + 1/y)$ and $u_y = -(1/y^2) \cdot f'(x^2 + 1/y)$, whence $u_x + 2xy^2u_y = 0$. □

In summary, the geometric method works nicely for any PDE of the form $a(x, y)u_x + b(x, y)u_y = 0$. It reduces the solution of the PDE to the solution of the ODE $dy/dx = b(x, y)/a(x, y)$. If the ODE can be solved, so can the PDE. Every solution of the PDE is constant on the solution curves of the ODE.

Moral Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called *initial* or *boundary* conditions. We shall encounter these conditions throughout the book.

EXERCISES

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)