EXERCISES

1. Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.
2. Which of the following operators are linear?
   (a) $\mathscr{L}u = u_x + xu_y$
   (b) $\mathscr{L}u = u_x + uu_y$
   (c) $\mathscr{L}u = u_x + u_y^2$
   (d) $\mathscr{L}u = u_x + u_y + 1$
   (e) $\mathscr{L}u = \sqrt{1+x^2}\,(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$
3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$
4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.
5. Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.
   (a) The vectors with $b = 0$.
   (b) The vectors with $b = 1$.
   (c) The vectors with $ab = 0$.
   (d) All the linear combinations of the two vectors [1, 1, 0] and [2, 0, 1].
   (e) All the vectors such that $c - a = 2b$.
6. Are the three vectors [1, 2, 3], [−2, 0, 1], and [1, 10, 17] linearly dependent or independent? Do they span all vectors or not?
7. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?
8. Find a vector that, together with the vectors [1, 1, 1] and [1, 2, 1], forms a basis of $\mathbb{R}^3$.
9. Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?
10. Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.
11. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.