Homework #11 Answers and Hints (MATH4052 Partial Differential Equations)

Xiaoyu Wei*

December 1, 2016

Problem 1. *(Page 196, Q1). Find the one-dimensional Green's function for the interval* $(0, l)$*. The three properties defining it can be restated as follows.*

1. *It solves* $G''(x) = 0$ *for* $x \neq x_0$ *("harmonic").*
2. $G(0) = G(l) = 0$.
3. $G(x)$ *is continuous at* x_0 *and* $G(x) + \frac{1}{2}|x - x_0|$ *is harmonic at* x_0.

Solution. It is clear from the harmonic condition that $G(x)$ is a piecewise linear function, whose derivative is continuous in $(0, x_0)$ and (x_0, l). Then, since the singularity at x_0 should be such that

$$G(x) + \frac{1}{2}|x - x_0|$$

is harmonic at x_0, it can be written as

$$G(x) = H(x) - \frac{1}{2}|x - x_0|,$$

where $H(x)$ is a harmonic (linear) function over $(0, l)$.

Lastly, to satisfy the boundary conditions,

$$0 = G(0) = H(0) - \frac{1}{2}|x_0|,$$
$$0 = G(l) = H(l) - \frac{1}{2}|l - x_0|.$$

From the equations above we can solve

$$H(x) = \frac{|l - x_0| - |x_0|}{2l}x + \frac{1}{2}|x_0|.$$

*For any issues, email me at xweiaf@connect.ust.hk.

Therefore,

$$G(x,x_0) = -\frac{1}{2}|x-x_0| + \frac{|l-x_0|-|x_0|}{2l}x + \frac{1}{2}|x_0|,$$

using the fact that $0 \le x_0 \le l$,

$$G(x,x_0) = -\frac{1}{2}|x-x_0| + \frac{l-2x_0}{2l}x + \frac{1}{2}x_0,$$

which is a piecewise linear function in x,

$$G(x,x_0) = \begin{cases} \frac{l-x_0}{l}x, & x \in [0,x_0], \\ -\frac{x_0}{l}x + x_0, & x \in [x_0,l]. \end{cases}$$

□

Problem 2. *(Page 196, Q6).*

1. *Find the Green's function for the half-plane* $\{(x,y) : y > 0\}$.
2. *Use it to solve the Dirichlet problem in the half-plane with boundary values* $h(x)$.
3. *Calculate the solution with* $u(x,0) = 1$.

Solution. We first state the three conditions that defined the Green's function $G(x,y)$ of $-\Delta$ at the point (x_0,y_0) on the half-plane:

1. $G_{xx} + G_{yy} = 0$ for $y > 0$ and $(x,y) \neq (x_0,y_0)$.
2. $G(x,0) = 0$.
3. $H(x,y) = G(x,y) - \frac{1}{2\pi}\ln\rho$, where $\rho^2 = (x-x_0)^2 + (y-y_0)^2$, is harmonic everywhere in the half-plane.

Using the reflection method, let $(x_0^\star, y_0^\star) = (x_0, -y_0)$ to be the reflected source point, and $\rho^\star = \sqrt{(x-x_0^\star)^2 + (y-y_0^\star)^2}$. Then

$$G(x,y) = \frac{1}{2\pi}\ln\rho - \frac{1}{2\pi}\ln\rho^\star.$$

Therefore,

$$\begin{aligned} u(x_0,y_0) &= -\int_{x=-\infty}^{+\infty} h(x)\frac{\partial G}{\partial y}(x,0)dx \\ &= -\frac{1}{2\pi}\int_{x=-\infty}^{+\infty} h(x)\frac{\partial(\ln\rho - \ln\rho^\star)}{\partial y}(x,0)dx \\ &= -\frac{1}{4\pi}\int_{x=-\infty}^{+\infty} h(x)\left[\frac{-2y_0 - 2y_0}{(x-x_0)^2 + y_0^2}\right]dx \\ &= \frac{1}{\pi}\int_{x=-\infty}^{+\infty} h(x)\left[\frac{y_0}{(x-x_0)^2 + y_0^2}\right]dx. \end{aligned}$$

When taking $h(x) = 1$, we have

$$\begin{aligned}
u(x_0, y_0) &= \frac{1}{\pi} \int_{x=-\infty}^{+\infty} \frac{y_0}{(x-x_0)^2 + y_0^2} dx \\
&= \frac{1}{\pi} \int_{x=-\infty}^{+\infty} \frac{1}{(\frac{x-x_0}{y_0})^2 + 1} \frac{1}{y_0} dx \\
&= \frac{1}{\pi} \int_{x=-\infty}^{+\infty} \frac{1}{(\frac{x-x_0}{y_0})^2 + 1} d\frac{x-x_0}{y_0} \\
&= 1,
\end{aligned}$$

which is what is expected when the boundary condition is a constant.

□

Problem 3. *(Page 197, Q9). Find the Green's function for the tilted half-space* $T = \{(x, y, z) : ax + by + cz > 0\}$. *(Hint : Either do it from scratch by reflecting across the tilted plane, or change variables in the double integral (3)*

$$0 = \iint_{bdy\ D} \left(u\frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H \right) dS$$

using a linear transformation.)

Solution. Let $(x^\star, y^\star, z^\star)$ be the reflection point of $(x, y, z) \in T$ with respect to ∂T, then $\exists \lambda \neq 0$, s.t.

$$\begin{aligned}
x^\star &= x + \lambda a \\
y^\star &= y + \lambda b \\
z^\star &= z + \lambda c \\
(x^\star)^2 + (y^\star)^2 + (z^\star)^2 &= x^2 + y^2 + z^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(x + \lambda a)^2 + (y + \lambda b)^2 + (z + \lambda c)^2 = x^2 + y^2 + z^2 \\
&2(ax + by + cz) + \lambda(a^2 + b^2 + c^2) = 0 \\
&\qquad\qquad \lambda = -2\frac{ax + by + cz}{a^2 + b^2 + c^2}.
\end{aligned}$$

It is clear that given $ax + by + cz > 0$,

$$\begin{aligned}
ax^\star + by^\star + cz^\star &= ax + by + cz + \lambda(a^2 + b^2 + c^2) \\
&= -(ax + by + cz) < 0.
\end{aligned}$$

Now, using the fundamental solution

$$F(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\rho},$$

where $\rho^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$, the Green's function of the tilted half-space is

$$G(x,y,z;x_0,y_0,z_0) = \frac{1}{4\pi\rho^\star} - \frac{1}{4\pi\rho},$$

with $(\rho^\star)^2 = (x-x_0^\star)^2 + (y-y_0^\star)^2 + (z-z_0^\star)^2$, and

$$\begin{aligned} x_0^\star &= x_0 - 2a\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}, \\ y_0^\star &= y_0 - 2b\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2}, \\ z_0^\star &= z_0 - 2c\frac{ax_0+by_0+cz_0}{a^2+b^2+c^2} \end{aligned}$$

□

Problem 4. *(Page 197, Q17).*

1. *Find the Green's function for the quadrant*

$$Q = \{(x,y) : x > 0, y > 0\}.$$

*(*Hint*: Either use the method of reflection or reduce to the half-plane problem by the transformation $(x,y) \mapsto (x^2-y^2, 2xy)$.)*

2. *Use your answer in the previous part to solve the Dirichlet problem*

$$u_{xx} + u_{yy} = 0 \text{ in } Q, \quad \begin{aligned} u(0,y) &= g(y) \text{ for } y > 0, \\ u(x,0) &= h(x) \text{ for } x > 0. \end{aligned}$$

Solution. Using the method of reflection, we reflect a point $(x_0,y_0) \in Q$ with respect to ∂Q to get its three images $(-x_0,y_0)$, $(x_0,-y_0)$, and $(-x_0,-y_0)$. The idea is to take linear superpositions of the fundamental solutions centered at these points to construct the Green's function.

Since the fundamental solution in 2D is

$$F(x,y;x_0,y_0) = \frac{1}{2\pi}\ln\rho,$$

where $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, the Green's function should be

$$G(x,y;x_0,y_0) = \frac{1}{2\pi}\ln\rho - \frac{1}{2\pi}\ln\rho_a - \frac{1}{2\pi}\ln\rho_b + \frac{1}{2\pi}\ln\rho_c,$$

where $\rho_a = \sqrt{(x+x_0)^2 + (y-y_0)^2}$, $\rho_b = \sqrt{(x-x_0)^2 + (y+y_0)^2}$, and $\rho_c = \sqrt{(x+x_0)^2 + (y+y_0)^2}$.

Now, given the boundary functions, we have

$$u(x_0,y_0) = -\int_0^\infty g(y)\frac{\partial G}{\partial x}(0,y)dy - \int_0^\infty h(x)\frac{\partial G}{\partial y}(x,0)dx.$$

Calculating the derivatives,

$$\frac{\partial G}{\partial x}(x,y;x_0,y_0) = \frac{1}{4\pi}\frac{2(x-x_0)}{\rho^2} - \frac{1}{4\pi}\frac{2(x+x_0)}{\rho_a^2} - \frac{1}{4\pi}\frac{2(x-x_0)}{\rho_b^2} + \frac{1}{4\pi}\frac{2(x+x_0)}{\rho_c^2}$$
$$\frac{\partial G}{\partial y}(x,y;x_0,y_0) = \frac{1}{4\pi}\frac{2(y-y_0)}{\rho^2} - \frac{1}{4\pi}\frac{2(y-y_0)}{\rho_a^2} - \frac{1}{4\pi}\frac{2(y+y_0)}{\rho_b^2} + \frac{1}{4\pi}\frac{2(y+y_0)}{\rho_c^2}$$

yielding

$$\frac{\partial G}{\partial x}(0,y;x_0,y_0) = -\frac{1}{\pi}\frac{x_0}{\rho^2} + \frac{1}{\pi}\frac{x_0}{\rho_c^2}$$
$$\frac{\partial G}{\partial y}(x,0;x_0,y_0) = -\frac{1}{\pi}\frac{y_0}{\rho^2} + \frac{1}{\pi}\frac{y_0}{\rho_c^2}$$

Then,

$$\begin{aligned} u(x_0,y_0) &= \int_0^\infty g(y)\left(-\frac{1}{\pi}\frac{x_0}{\rho^2} + \frac{1}{\pi}\frac{x_0}{\rho_c^2}\right)dy - \int_0^\infty h(x)\left(-\frac{1}{\pi}\frac{y_0}{\rho^2} + \frac{1}{\pi}\frac{y_0}{\rho_c^2}\right)dx \\ &= -\int_0^\infty g(y)\left(\frac{1}{\pi}\frac{x_0}{x_0^2+(y-y_0)^2} - \frac{1}{\pi}\frac{x_0}{x_0^2+(y+y_0)^2}\right)dy + \\ &\quad \int_0^\infty h(x)\left(\frac{1}{\pi}\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{1}{\pi}\frac{y_0}{(x+x_0)^2+y_0^2}.\right)dx \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,y) = &-\int_0^\infty xg(\eta)\left[\frac{1}{(y-\eta)^2+x^2} - \frac{1}{(y+\eta)^2+x^2}\right]\frac{d\eta}{\pi} + \\ &\int_0^\infty yh(\xi)\left[\frac{1}{(x-\xi)^2+y^2} - \frac{1}{(x+\xi)^2+y^2}.\right]\frac{d\xi}{\pi}. \end{aligned}$$

Remark 1. *Note should be taken for the signs of the integrals consisting of $u(x,y)$.*

□