# Homework #10 Answers and Hints
# (MATH4052 Partial Differential Equations)

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**Problem 1.** *(Page 184, Q2). Prove the uniqueness up to constants of the Neumann problem using the energy method.*

*Solution.* Suppose that there exist two solutions $u_1$ and $u_2$, then $w = u_1 - u_2$ solves the homogeneous Neumann problem

$$\Delta w = 0 \quad in \quad D \tag{1}$$

$$\frac{\partial w}{\partial n} = 0 \quad on \quad \text{bdy } D. \tag{2}$$

Test the equation with $w$, and apply Green's first identity

$$\int_{\partial D} w \frac{\partial w}{\partial n} dS = \int_D \nabla w \cdot \nabla w d\mathbf{x} + \int_D w \Delta w d\mathbf{x}.$$

Therefore, $w$ has vanishing energy

$$E[w] = \int_D \nabla w \cdot \nabla w d\mathbf{x} = 0.$$

Since the integrand is nonnegative, it follows from the vanishing theorem that $|\nabla w|^2 = 0$ in $D$, that is, $\nabla w = \mathbf{0}$ in $D$, and we deduce that $w$ is a constant in $D$, which proves the uniqueness up to constants.

□

**Problem 2.** *(Page 184, Q3). Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.*

*Solution.* Similar to the previous one, suppose that there exist two solutions $u_1$ and $u_2$, then $w = u_1 - u_2$ solves the homogeneous Robin problem

$$\Delta w = 0 \quad in \quad D$$

$$\frac{\partial w}{\partial n} + a(\mathbf{x})w(\mathbf{x}) = 0 \quad on \quad \partial D.$$

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Test the equation with $w$, and apply Green's first identy

$$\int_{\partial D} w\frac{\partial w}{\partial n}dS = \int_D \nabla w \cdot \nabla w d\mathbf{x} + \int_D w\Delta w d\mathbf{x}.$$

Therefore, since $a > 0$, the above equation yields

$$-\int_{\partial D} aw^2 dS = \int_D \nabla w \cdot \nabla w d\mathbf{x} = 0.$$

Consequently, $w = 0$ in $D$, which proves the uniqueness of the given Robin problem.

□

**Problem 3.** *(Page 187, Q1). Derive the representation formula for harmonic functions in two dimensions:*

$$u(\mathbf{x}_0) = \frac{1}{2\pi}\int_{bdyD}\left[u(\mathbf{x})\frac{\partial}{\partial n}(\log|\mathbf{x}-\mathbf{x}_0|) - \frac{\partial u}{\partial n}\log|\mathbf{x}-\mathbf{x}_0|\right]ds.$$

*Solution.* First of all, using polar coordinates (setting $\mathbf{x}_0$ to be the origin) , it is easy to verify that in two dimensions, the function

$$G(r,\theta) = \frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}_0| = \frac{1}{2\pi}\ln r$$

is harmonic for $r \neq 0$ (i.e., $\mathbf{x} \neq \mathbf{x}_0$).

Then, using the same procedure as in the derivation of higher dimensions, one obtains the representation formula in two dimensions. □

**Problem 4.** *(Page 187, Q2). Let $\phi(\mathbf{x})$ be any $\mathbf{C}^2$ function defined on all of three-dimensional space that vanishes outside some sphere. Show that*

$$\phi(\mathbf{0}) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

*The integration is taken over the region where $\phi(\mathbf{x})$ is not zero.*

*Solution.* First, define the region $D$ such that $\text{supp}(\phi) \subset D$. Then, let $D_\epsilon = D \setminus U(\mathbf{0},\epsilon)$ being $D$ with a unit ball centered at $\mathbf{0}$ with radius $\epsilon$ excluded, where $\epsilon$ is small enough such that $\partial U(\mathbf{0},\epsilon) \cap \partial D = \emptyset$. Via Green's second equality, denoting $g(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$,

$$\int_{D_\epsilon} g\Delta\phi - \phi\Delta g d\mathbf{x} = \int_{\partial D_\epsilon} g\frac{\partial\phi}{\partial n} - \phi\frac{\partial g}{\partial n}dS,$$

thus

$$\begin{aligned}
\int_{D_\epsilon} g\Delta\phi d\mathbf{x} &= \int_{r=\epsilon} g\frac{\partial\phi}{\partial n} - \phi\frac{\partial g}{\partial n} dS, \\
&= -\frac{1}{\epsilon}\int_{r=\epsilon} \frac{\partial\phi}{\partial r} dS - \frac{1}{\epsilon^2}\int_{r=\epsilon} \phi dS \\
&= -4\pi\epsilon\frac{\partial\bar{\phi}}{\partial r} - 4\pi\bar{\phi} \\
&\longrightarrow -4\pi\phi(\mathbf{0}), \quad \text{as } \epsilon \longrightarrow 0.
\end{aligned}$$

Therefore,

$$\phi(\mathbf{0}) = -\int_D \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

□