Homework #09 Answers and Hints
(MATH4052 Partial Differential Equations)

Xiaoyu Wei*

November 14, 2016

Problem 1. *(Page 172, Q1). Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions:*

1. *Find the maximum value of u in $\bar{D}$.*
2. *Calculate the value of u at the origin.*

Solution. We have

1. From maximum principle,

$$\max_D u(r,\theta) = \max_{\partial D} u = \max_\theta (3\sin 2\theta + 1) = 4.$$

2. From mean value theorem, denote the value of u at origin as u_0,

$$\begin{aligned} u_0 &= \frac{1}{4\pi}\int_{\theta=0}^{2\pi}(3\sin 2\theta + 1)2d\theta \\ &= 1. \end{aligned}$$

□

Problem 2. *(Page 172, Q2). Solve $u_{xx} + u_{yy} = 0$ in the disk $D = \{r < a\}$ with the boundary condition*

$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$

Solution. Using Poisson's formula,

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{1 + 3\sin\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2}\frac{d\phi}{2\pi}.$$

*For any issues, email me at xweiaf@connect.ust.hk.

First of all, when $r=0$, we have the value of u at the origin

$$u(r=0) = \frac{1}{2\pi a}\int_0^{2\pi}(1+3\sin\theta)a d\theta = 1.$$

From now on we assume $0<r<1$. Let $s=\frac{r}{a}$,

$$u(r,\theta) = (1-s^2)\int_0^{2\pi}\frac{1+3\sin\phi}{1-2s\cos(\theta-\phi)+s^2}\frac{d\phi}{2\pi}.$$

Change of variable $\phi \leftarrow (\theta-\phi)/2$,

$$\begin{aligned}
u(r,\theta) &= (1-s^2)\int_{\theta/2}^{\theta/2+\pi}\frac{1+3\sin(\theta-2\phi)}{1-2s\cos 2\phi+s^2}\frac{d\phi}{\pi}\\
&= (1-s^2)\int_0^{\pi}\frac{(1+3\sin\theta)\cos^2\phi+(1-3\sin\theta)\sin^2\phi-6\cos\theta\sin\phi\cos\phi}{(1+s)^2\sin^2\phi+(1-s)^2\cos^2\phi}\frac{d\phi}{\pi}\\
&= (1-s^2)\int_{-\pi/2}^{\pi/2}\frac{(1+3\sin\theta)+(1-3\sin\theta)\tan^2\phi-6\cos\theta\tan\phi}{(1+s)^2\tan^2\phi+(1-s)^2}\frac{d\phi}{\pi}.
\end{aligned}$$

Change of variable $t \leftarrow \tan\phi$,

$$\begin{aligned}
u(r,\theta) &= (1-s^2)\int_{-\infty}^{\infty}\frac{(1-3\sin\theta)t^2-(6\cos\theta)t+(1+3\sin\theta)}{(1+s)^2t^2+(1-s)^2}\frac{1}{t^2+1}\frac{dt}{\pi}\\
&= \frac{1-s}{1+s}\int_{-\infty}^{\infty}\frac{(1-3\sin\theta)t^2-(6\cos\theta)t+(1+3\sin\theta)}{t^2+p^2}\frac{1}{t^2+1}\frac{dt}{\pi},
\end{aligned}$$

where $p=\frac{1-s}{1+s}=\frac{a-r}{a+r}\in(0,1)$. Noting that $\frac{1+p^2}{1-p^2}=1$.

One way to calculate this integral is to directly use residues from complex analysis. To do that, one needs to compute the residues at two poles i and pi. To simplify the computation, rational function decompositon may as well be applied.

Assume the decomposition for the integrand of the form

$$\frac{(1-3\sin\theta)t^2-(6\cos\theta)t+(1+3\sin\theta)}{t^2+p^2}\frac{1}{t^2+1} = \frac{At+B}{t^2+p^2}+\frac{Ct+D}{t^2+1},$$

solving

$$\begin{aligned}
A &= -\frac{6}{1-p^2}\cos\theta,\\
B &= 1-3\sin\theta+\frac{6}{1-p^2}\sin\theta,\\
C &= \frac{6}{1-p^2}\cos\theta,\\
D &= -\frac{6}{1-p^2}\sin\theta.
\end{aligned}$$

Utilizing $\int_{-\infty}^{\infty} \frac{At+B}{t^2+y^2} dt = \frac{\pi}{y} B + iA\pi$, (which can be easily calculated using residues from complex analysis, alternatively, see the remark below), we have

$$\begin{aligned} u(r,\theta) &= p(\frac{1}{p} B + iA + D + iC) \\ &= B + pD + ip(A+C) \\ &= B + pD. \end{aligned}$$

Therefore, after simplification,

$$u(r,\theta) = 1 + \frac{3r}{a} \sin\theta.$$

Remark 1. *Another way to calculate the integral is to first consider a symmetric bounded region $(-M, M)$, such that*

$$\begin{aligned} u_M(r,\theta) &= \frac{1-s}{1+s} \int_{-M}^{M} \frac{(1-3\sin\theta)t^2 - (6\cos\theta)t + (1+3\sin\theta)}{t^2+p^2} \frac{1}{t^2+1} \frac{dt}{\pi} \\ &= \frac{1-s}{1+s} \int_{-M}^{M} \frac{At+B}{t^2+p^2} + \frac{Ct+D}{t^2+1} \frac{dt}{\pi} \\ &= \frac{1-s}{1+s} \int_{-M}^{M} \frac{At+B}{t^2+p^2} \frac{dt}{\pi} + \frac{1-s}{1+s} \int_{-M}^{M} \frac{Ct+D}{t^2+1} \frac{dt}{\pi}, \end{aligned}$$

Due to the symmetry of the interval, the odd parts do not contribute to the integal. While the even parts are easy to integrate. After obtaining u_M, let $M \to \infty$, and one still gets the same results

$$\begin{aligned} u(r,\theta) &= \lim_{M\to\infty} u_M \\ &= B + pD. \end{aligned}$$

□

Problem 3. *(Page 175, Q1). Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \longrightarrow \infty$.*

Solution. Via Possion's formula (exterior version),

$$u(r,\theta) = (r^2 - a^2) \int_0^{2\pi} \frac{1 + 3\sin\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2} \frac{d\phi}{2\pi},$$

which can be obtained from the interior solution by changing the variables $r \leftarrow r^{-1}$, and $a \leftarrow a^{-1}$.

Therefore, utilizing solutions from the previous problem,

$$u(r,\theta) = 1 + \frac{3a}{r} \sin\theta.$$

□

Problem 4. *(Page 176, Q10). Solve* $u_{xx}+u_{yy}=0$ *in the quarter-disk* $\{x^2+y^2<a^2, x>0, y>0\}$ *with the following BCs:*

$$u=0 \quad \text{on } x=0 \text{ and on } y=0 \quad \text{and} \quad \frac{\partial u}{\partial r}=1 \quad \text{on } r=a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

Solution. The equation written in polar coordinate is

$$0=u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}.$$

Adopting separation of variables,

$$u(r,\theta)=R(r)\Theta(\theta),$$

we have

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta'',$$

resulting in the following eigenvalue problem

$$\Theta''+\lambda\Theta=0,$$
$$r^2R''+rR'-\lambda R=0,$$

subject to the following boundary conditions

$$\Theta(0)=\Theta\left(\frac{\pi}{2}\right)=0,$$
$$R(0)=0.$$

The resulting eigenpairs are $(n\geq 1)$

$$\lambda_n=4n^2,$$
$$\Theta_n(\theta)=\sin(2n\theta),$$
$$R_n(r)=r^{2n}.$$

Now let

$$u(r,\theta)=\sum_{n=1}^{\infty}A_n r^{2n}\sin(2n\theta),$$

and choose A_n to satisfy the inhomogeneous boundary condition

$$\begin{aligned}1&=u_r(a,\theta)\\&=\sum_{n=1}^{\infty}A_n 2na^{2n-1}\sin(2n\theta),\end{aligned}$$

taking inner product with $\sin(2m\theta), m \geq 1$ on both sides,

$$\int_0^{\frac{\pi}{2}} sin(2m\theta)d\theta = 2ma^{2m-1}A_m \int_0^{\frac{\pi}{2}} sin^2(2m\theta)d\theta$$
$$\frac{1}{m}\sin^2\left(\frac{\pi m}{2}\right) = \frac{\pi m a^{2m-1}}{2}A_m$$
$$A_m = \frac{2}{\pi m^2 a^{2m-1}}\sin^2\left(\frac{\pi m}{2}\right).$$

Therefore, the first two nonzero terms are

$$(m=1): \quad \frac{2r^2}{\pi a}\sin 2\theta,$$
$$(m=3): \quad \frac{2r^6}{9\pi a^5}\sin 6\theta.$$

□