Homework #08 Answers and Hints (MATH4052 Partial Differential Equations)

Xiaoyu Wei*

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Problem 1. *(Page 160, Q6). Solve $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u(x, y)$ vanishing on both parts of the boundary $r = a$ and $r = b$.*

Solution. Introduce a transformation

$$x = r\cos\theta, \quad y = r\sin\theta,$$

taking the equation into polar coordinates, and assume that the solution only depends on r, $u = u(r)$ (via uniqueness of the Dirichlet problem, if we can indeed find one such solution, then it should be the only one). The PDE can be reduced to an ODE with boundary values

$$\begin{aligned} u'' + \frac{1}{r}u' &= 1, \\ u(a) = u(b) &= 0. \end{aligned}$$

Recall that this is an inhomogeneous Euler-Cauchy equation, to solve it, general solution to its homogeneous counterpart is needed

$$r^2 v'' + rv' = 0,$$

which can be solved by looking for solutions of the form $v(r) = r^m$. In doing so one obtains a quadratic equation for m

$$\begin{aligned} m(m-1) + m &= 0 \\ m^2 &= 0, \end{aligned}$$

solving only one real root $m = 0$. Then the general solution to the homogeneous equation is $v(r) = Ar^0 + Br^0 \ln(r) = A + B\ln(r)$. Then, to find a particular solution to the inhomogeneous equation, use the method of undetermined coefficients, letting $u(r) = a_2 r^2 + a_1 r + a_0$,

$$2a_2 + 2a_2 + \frac{1}{r}a_1 = 1,$$

*For any issues, email me at xweiaf@connect.ust.hk.

then one solution is $u(r) = \frac{1}{4}r^2$. Therefore, the general solution to the inhomogeneous ODE is

$$u(r) = \frac{1}{4}r^2 + A + B\ln(r).$$

Matching boundary condition at $r = a, r = b$,

$$0 = \frac{1}{4}a^2 + A + B\ln(a),$$
$$0 = \frac{1}{4}b^2 + A + B\ln(b),$$

solving

$$A = \frac{b^2\ln(a) - a^2\ln(b)}{4[\ln(b) - \ln(a)]},$$
$$B = -\frac{b^2 - a^2}{4[\ln(b) - \ln(a)]}.$$

Therefore,

$$u(r) = \frac{1}{4}r^2 + \frac{b^2\ln(a) - a^2\ln(b)}{4[\ln(b) - \ln(a)]} - \frac{b^2 - a^2}{4[\ln(b) - \ln(a)]}\ln(r).$$

□

Problem 2. *(Page 160, Q9). A spherical shell with inner radius* 1 *and outer radius* 2 *has a steady-state temperature distribution. Its inner boundary is held at* 100°C. *Its outer boundary satisfies* $\partial u/\partial r = -\gamma < 0$, *where* γ *is a constant.*

1. *Find the temperature.*
   (Hint : The temperature depends only on the radius.)
2. *What are the hottest and coldest temperatures?*
3. *Can you choose* γ *so that the temperature on its outer boundary is* 20°C*?*

Solution. Assume that the temperature depends only on the radius r, then we have the heat equation in 3D without heat source as an ODE

$$u'' + \frac{2}{r}u' = 0,$$
$$u(1) = 100, \quad u'(2) = -\gamma.$$

The general solution is again sought in the form of $u(r) = r^m$, yielding

$$m(m-1) + 2m = 0$$
$$m^2 + m = 0$$
$$\text{solving} \quad m_1 = 0, \quad m_2 = -1,$$

thus

$$u(r) = A + \frac{B}{r}.$$

Matching boundary conditions,

$$\begin{aligned} 100 &= u(1) = A + B, \\ -\gamma &= u'(2) = -\frac{B}{4}, \end{aligned}$$

solving

$$\begin{aligned} A &= 100 - 4\gamma, \\ B &= 4\gamma. \end{aligned}$$

Therefore **the temperature** is,

$$u(r) = (100 - 4\gamma) + \frac{4\gamma}{r} = 4\gamma \left(\frac{1}{r} - 1 \right) + 100.$$

It is obvious that $u(r)$ is monotonically decreasing in r as $\gamma > 0$; therefore, the **hottest** temperature is $u(1) = 100$, and the **coldest** temperature is $u(2) = 100 - 2\gamma$.

Lastly, to have $u(2) = 100 - 2\gamma = 20$, simply letting $\gamma = 40$ will do.

$\square$

Problem 3. *(Page 164 to Page 165, Q1). Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:*

$$\begin{aligned} u_x &= -a \quad \text{on } x = 0 \qquad & u_x &= 0 \quad \text{on } x = a \\ u_y &= b \quad \text{on } y = 0 \qquad & u_y &= 0 \quad \text{on } y = b. \end{aligned}$$

(Hint : Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

Solution. Taking advantage of the hint, we look for solutions of the form

$$u(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1 x + b_2 y + c,$$

thus

$$\begin{aligned} u_x(x, y) &= 2a_{11}x + 2a_{12}y + b_1, \\ u_y(x, y) &= 2a_{12}x + 2a_{22}y + b_2. \end{aligned}$$

For such u to be harmonic, we have

$$\begin{aligned} 0 &= u_{xx} + u_{yy} \\ &= 2a_{11} + 2a_{22}. \end{aligned}$$

And plugging in boundary condition, we have $a_{11} = \frac{1}{2}, a_{12} = 0, a_{22} = -\frac{1}{2}, b_1 = -a, b_1 = a$ and no constraints on c. Therefore, via uniqueness of Neumann problems up to a constant, we have the solution to be

$$u(x, y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + c, \quad c \in \mathbb{R}.$$

□

Problem 4. *(Page 165, Q4). Find the harmonic function in the square* $\{0 < x < 1, 0 < y < 1\}$ *with the boundary conditions* $u(x, 0) = x$, $u(x, 1) = 0$, $u_x(0, y) = 0$, $u_x(1, y) = y^2$.

Solution. Taking advantage of linearity, we can write the solution into two parts

$$u(x, y) = U(x, y) + V(x, y),$$

where both U, V are harmonic, and they satisfy the following boundary conditions

$$\begin{aligned} &U(x, 0) = x, \quad U(x, 1) = 0, \quad U_x(0, y) = 0, \quad U_x(1, y) = 0, \\ &V(x, 0) = 0, \quad V(x, 1) = 0, \quad V_x(0, y) = 0, \quad V_x(1, y) = y^2. \end{aligned}$$

Then each part can be solved utilizing the fact that the domain is a square. For example, if we consider harmonic functions of the form

$$W(x, y) = X(x)Y(y),$$

then

$$0 = W_{xx} + W_{yy} = X''Y + XY'',$$

and we have

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} := \lambda.$$

Therefore, we have

$$\begin{aligned} X'' &= \lambda X, \\ Y'' &= -\lambda Y. \end{aligned}$$

We can use either one of the ODEs above, together with proper homogeneous boundary conditions, to form an eigenvalue problem.

For U, we can use

$$X'' = \lambda X, \quad \text{with} \quad X'(0) = 0, \quad X'(1) = 0,$$

and for V, we can use

$$Y'' = -\lambda Y, \quad \text{with} \quad Y(0) = 0 \quad Y(1) = 0.$$

Now we solve for U, V separately.

1. **Solve for** $U(x,y)$. From the first eigenvalue problem given above, we have

$$\lambda_n = -n^2\pi^2,$$
$$X_n(x) = A\cos(n\pi x).$$

Using this to solve an ODE involving $Y(y)$ and the homogeneous part of its boundary conditions

$$Y_n'' = -\lambda_n Y_n, \quad \text{with} \quad Y(1) = 0,$$

yields

$$Y_0(y) = C_0(1-y), \quad C_0 \in \mathbb{R}$$
$$Y_n(y) = C_n\left(e^{n\pi y} - e^{2n\pi - n\pi y}\right), \quad n \geq 1, \quad C_n \in \mathbb{R}.$$

Now we write $U(x,y)$ as a double Fourier series

$$\begin{aligned} U(x,y) &= \sum_{n=0}^{\infty} X_n(x)Y_n(y) \\ &= F_0(1-y) + \sum_{n=1}^{\infty} F_n\cos(n\pi x)\left(e^{n\pi y} - e^{2n\pi - n\pi y}\right). \end{aligned}$$

Apply the inhomogeneous boundary condition on the series,

$$\begin{aligned} x &= U(x,0) \\ &= F_0 + \sum_{n=1}^{\infty} F_n\cos(n\pi x)\left(1 - e^{2n\pi}\right). \end{aligned}$$

Take inner product with $\cos(m\pi x)$ on both sides, $m \in \mathbb{N}$, yielding

$$\begin{aligned} \langle x, 1\rangle &= F_0\langle 1, 1\rangle \\ F_0 &= \frac{1}{2}, \\ \langle x, \cos(m\pi x)\rangle &= F_m\left(1 - e^{2m\pi}\right)\langle\cos(m\pi x), \cos(m\pi x)\rangle \\ F_m &= \frac{1}{1 - e^{2m\pi}}\frac{\langle x, \cos(m\pi x)\rangle}{\langle\cos(m\pi x), \cos(m\pi x)\rangle} \\ &= \frac{2}{1 - e^{2m\pi}}\frac{(-1)^m - 1}{\pi^2 m^2}, \quad m \geq 1. \end{aligned}$$

So

$$U(x,y) = \frac{1}{2}(1-y) + \sum_{n=1}^{\infty}\frac{-4\cos\left[(2n-1)\pi x\right]}{\pi^2(2n-1)^2\left(1 - e^{(4n-2)\pi}\right)}\left[e^{(2n-1)\pi y} - e^{(2n-1)\pi(2-y)}\right].$$

2. **Solve for** $V(x,y)$. From the second eigenvalue problem above, we have

$$\lambda_n = n^2\pi^2,$$
$$Y_n(y) = A\sin(n\pi y).$$

Using this to solve the ODE involving $X(x)$ subject to the homogeneous part of its boundary conditions

$$X_n'' = \lambda_n X_n, \quad \text{with} \quad X'(0) = 0,$$

yields

$$X_n(x) = D\left(e^{n\pi x} + e^{-n\pi x}\right).$$

Now we write $V(x,y)$ also as a double Fourier series

$$\begin{aligned} V(x,y) &= \sum_{n=1}^{\infty} X_n(x)Y_n(y) \\ &= \sum_{n=1}^{\infty} G_n\left(e^{n\pi x} + e^{-n\pi x}\right)\sin(n\pi y) \end{aligned}$$

Apply the inhomogeneous boundary condition on the series,

$$\begin{aligned} y^2 &= V(1,y) \\ &= \sum_{n=1}^{\infty} G_n\left(e^{n\pi} + e^{-n\pi}\right)\sin(n\pi y). \end{aligned}$$

Take inner product with $\sin(m\pi y)$ on both sides, $m \in \mathbb{N}$, yielding

$$\begin{aligned} \left\langle y^2, \sin(m\pi y)\right\rangle &= G_m\left(e^{m\pi} + e^{-m\pi}\right)\langle\sin(m\pi y), \sin(m\pi y)\rangle \\ G_m &= \frac{2}{e^{m\pi} + e^{-m\pi}} \frac{(-1)^m(2-\pi^2m^2) - 2}{\pi^3m^3}. \end{aligned}$$

So

$$V(x,y) = \sum_{n=1}^{\infty} 2\frac{(-1)^m(2-\pi^2m^2) - 2}{\pi^3m^3} \frac{e^{n\pi x} + e^{-n\pi x}}{e^{n\pi} + e^{-n\pi}} \sin(n\pi y).$$

In conclusion,

$$\begin{aligned} u(x,y) &= U(x,y) + V(x,y) \\ &= \frac{1}{2}(1-y) + \sum_{n=1}^{\infty} \frac{-4\cos\left[(2n-1)\pi x\right]}{\pi^2(2n-1)^2\left(1 - e^{(4n-2)\pi}\right)}\left[e^{(2n-1)\pi y} - e^{(2n-1)\pi(2-y)}\right] \\ &\quad + \sum_{n=1}^{\infty} 2\frac{(-1)^m(2-\pi^2m^2) - 2}{\pi^3m^3} \frac{e^{n\pi x} + e^{-n\pi x}}{e^{n\pi} + e^{-n\pi}} \sin(n\pi y). \end{aligned}$$

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