Homework #07 Answers and Hints
(MATH4052 Partial Differential Equations)

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Problem 1. *(Page 111, Q2). Let* $\phi(x) \equiv x^2$ *for* $0 \le x \le 1 = l$.

1. *Calculate its Fourier sine series.*
2. *Calculate its Fourier cosine series.*

Solution. Calculate via definitions.

1. Let
$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$
in the interval $(0, l)$. Here $l = 1$, and (note that $\sin(m\pi) = 0$ since m is integer)
$$\begin{aligned}
A_m &= \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)dx \\
&= 2\int_0^1 x^2 \sin(m\pi x)\,dx \\
&= -2\int_0^1 \frac{x^2}{m\pi} d\cos(m\pi x) \\
&= -2\frac{1}{m\pi}\cos(m\pi) + 2\int_0^1 \frac{2x}{m\pi}\cos(m\pi x)\,dx \\
&= -\frac{2\cos(m\pi)}{m\pi} + 2\int_0^1 \frac{2x}{m^2\pi^2} d\sin(m\pi x) \\
&= -\frac{2\cos(m\pi)}{m\pi} + 2\frac{2}{m^2\pi^2}\sin(m\pi) - 2\int_0^1 \frac{2}{m^2\pi^2}\sin(m\pi x)\,dx \\
&= -\frac{2\cos(m\pi)}{m\pi} - \frac{4}{m^2\pi^2}\int_0^1 \sin(m\pi x)\,dx \\
&= -\frac{2\cos(m\pi)}{m\pi} + \frac{4}{m^3\pi^3}\left[\cos(m\pi) - 1\right],
\end{aligned}$$

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thus the Fourier sine series

$$\phi(x) \sim \sum_{n=1}^{\infty} \frac{(4 - 2n^2\pi^2)\cos(n\pi) - 4}{n^3\pi^3} \sin(n\pi x).$$

2. Let

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

in the interval $(0, l)$. Here $l = 1$, and if $m = 0$,

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{0\pi x}{l}\right) dx \\ &= 2\int_0^1 x^2 dx \\ &= \frac{2}{3}, \end{aligned}$$

otherwise, if $m > 0$,

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right) dx \\ &= 2\int_0^1 x^2 \cos(m\pi x)\, dx \\ &= \frac{2}{m\pi}\int_0^1 x^2 d\sin(m\pi x) \\ &= \frac{2}{m\pi}\sin(m\pi) - \frac{2}{m\pi}\int_0^1 2x\sin(m\pi x)\, dx \\ &= \frac{4}{m^2\pi^2}\int_0^1 x d\cos(m\pi x) \\ &= \frac{4}{m^2\pi^2}\cos(m\pi) - \frac{4}{m^2\pi^2}\int_0^1 \cos(m\pi x)\, dx \\ &= \frac{4}{m^2\pi^2}\cos(m\pi) - \frac{4}{m^3\pi^3}\left[\sin(m\pi) - \sin(0)\right] dx \\ &= \frac{4}{m^2\pi^2}\cos(m\pi), \end{aligned}$$

thus the Fourier cosine series

$$\phi(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi)\cos(n\pi x).$$

□

Problem 2. *(Page 111, Q4). Find the Fourier cosine series of the function* $|\sin x|$ *in the interval* $(-\pi, \pi)$. *Use it to find the sums*

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Solution. Since $\phi(x) = |\sin x|$ is an even function, its full Fourier series collapses the Fourier cosine series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(nx),$$

where

$$\begin{aligned} A_0 &= \frac{2}{\pi}\int_0^{\pi} \phi(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(x)dx \\ &= \frac{4}{\pi}, \end{aligned}$$

and

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^{\pi} \phi(x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin(x+nx)+\sin(x-nx)dx \\ &= \frac{1}{\pi}\int_0^{\pi} \sin[(1+n)x]dx + \frac{1}{\pi}\int_0^{\pi} \sin[(1-n)x]dx, \end{aligned}$$

then, if $n = 1$,

$$\begin{aligned} A_1 &= \frac{1}{\pi}\int_0^{\pi} \sin(2x)dx + 0 \\ &= 0, \end{aligned}$$

and, if $n > 1$,

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{\pi} \sin[(1+n)x]dx + \frac{1}{\pi}\int_0^{\pi} \sin[(1-n)x]dx \\ &= -\frac{1}{\pi(1+n)}\int_0^{\pi} d\cos[(1+n)x] - \frac{1}{\pi(1-n)}\int_0^{\pi} d\cos[(1-n)x] \\ &= -\frac{(-1)^{n+1}-1}{\pi(1+n)} - \frac{(-1)^{n-1}-1}{\pi(1-n)} \\ &= \frac{(-1)^n+1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right). \end{aligned}$$

Therefore, only even n gives nonzero A_n, and we have

$$\begin{aligned}\phi(x) &= \frac{2}{\pi} + \sum_{m=1}^{\infty} \frac{-2}{\pi}\left(\frac{1}{2m+1} - \frac{1}{2m-1}\right)\cos(2mx)\\ &= \frac{2}{\pi} + \sum_{m=1}^{\infty} \frac{-2}{\pi}\frac{2}{4m^2-1}\cos(2mx).\end{aligned}$$

Take $x = 0$, we have

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{4m^2-1},$$

which yields

$$\sum_{m=1}^{\infty}\frac{1}{4m^2-1} = \frac{1}{2}.$$

Similarly, take $x = \frac{\pi}{2}$, and we have

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{4m^2-1},$$

giving

$$\sum_{m=1}^{\infty}\frac{(-1)^m}{4m^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

□

Problem 3. *(Page 134, Q1).* $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ *is a geometric series.*

1. *Does it converge pointwise in the interval* $-1 < x < 1$ *?*
2. *Does it converge uniformly in the interval* $-1 < x < 1$ *?*
3. *Does it converge in the* L^2 *sense in the interval* $-1 < x < 1$ *?*
   (Hint: You can compute its partial sums explicitly.)

Solution. Since it is a geometric series, define the partial sum

$$S_N(x) = \sum_{n=0}^{N}(-1)^n x^{2n}.$$

Then we have $S_0 = 1$, and

$$\begin{aligned}(-1)x^2 S_N(x) &= \sum_{n=0}^{N}(-1)^{n+1}x^{2(n+1)}\\ &= S_N + (-1)^{n+1}x^{2(n+1)} - 1\end{aligned}$$

Therefore,

$$S_N = \frac{(-1)^{n+1}x^{2(n+1)} - 1}{-x^2 - 1}.$$

Let $S = \lim_{N\to\infty} S_N = \frac{1}{x^2+1}$, then

1. (Pointwise convergence) $\forall x \in (-1, 1)$, we have

$$\lim_{N\to\infty} S_N(x) = S(x).$$

2. (Non-uniform convergence) $\forall \epsilon \in (0, 1)$, to have $|S_N(x) - S(x)| < \epsilon$, since

$$\begin{aligned} |S_N(x) - S(x)| &= \left| \frac{(-1)^{N+1}x^{2(N+1)}}{1 + x^2} \right| \\ &\geq \frac{1}{2} \left| x^{2(N+1)} \right|, \end{aligned}$$

it must be that

$$\begin{aligned} |x|^{2(N+1)} &< 2\epsilon \\ 2(N+1)\log|x| &< \log(2\epsilon) \\ N &> \frac{1}{2}\frac{\log(2\epsilon)}{\log|x|} - 1. \end{aligned}$$

For fixed ϵ, the right hand side goes to ∞ as $|x| \to 1$, so there does not exist such a uniform lower bound $N > N^\star$ that $|S_N(x) - S(x)| < \epsilon$ holds for $\forall x$.

3. (L^2 convergence) We have

$$\begin{aligned} ||S_N - S||_{L^2}^2 &= \int_{-1}^{1} |S_N(x) - S(x)|^2 \, dx \\ &= \int_{-1}^{1} \frac{x^{2(N+1)}}{1 + x^2} dx \\ &\leq \int_{-1}^{1} x^{2(N+1)} dx \\ &= \frac{(1 - (-1)^{2N+3})}{2N + 3} \\ &\leq \frac{2}{2N + 3}, \end{aligned}$$

therefore,

$$\lim_{N\to\infty} ||S_N - S||_{L^2} = 0.$$

□

Problem 4. *(Page 134, Q5). Let $\phi(x) = 0$ for $0 < x < 1$ and $\phi(x) = 1$ for $1 < x < 3$.*

1. *Find the first four nonzero terms of its Fourier cosine series explicitly.*
2. *For each x $(0 \le x \le 3)$, what is the sum of this series?*
3. *Does it converge to $\phi(x)$ in the L^2 sense? Why?*
4. *Put $x = 0$ to find the sum*

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \dots.$$

Solution. Consider the even extension of $\phi(x)$, its full Fourier series collapses to Fourier cosine series. What is more, it is continuous at $x = 0$, yielding strong convergence at this point.

1. Write

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{3}\right),$$

where

$$\begin{aligned} A_0 &= \frac{2}{3}\int_0^3 \phi(x)dx \\ &= \frac{2}{3}\int_1^3 1dx \\ &= \frac{4}{3}, \end{aligned}$$

and for $n \ge 1$,

$$\begin{aligned} A_n &= \frac{2}{3}\int_0^3 \phi(x)\cos(\frac{n\pi x}{3})dx \\ &= \frac{2}{3}\int_1^3 \cos(\frac{n\pi x}{3})dx \\ &= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right). \end{aligned}$$

Then the Fourier cosine series is

$$\begin{aligned} \phi(x) &\sim \frac{2}{3} - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi x}{3}\right) \\ &= \frac{2}{3} - \frac{2}{\pi}\frac{\sqrt{3}}{2}\cos\left(\frac{\pi x}{3}\right) - \frac{2}{2\pi}\frac{\sqrt{3}}{2}\cos\left(\frac{2\pi x}{3}\right) + \frac{2}{4\pi}\frac{\sqrt{3}}{2}\cos\left(\frac{4\pi x}{3}\right) + \dots \\ &= \frac{2}{3} - \frac{\sqrt{3}}{\pi}\cos\left(\frac{\pi x}{3}\right) - \frac{\sqrt{3}}{2\pi}\cos\left(\frac{2\pi x}{3}\right) + \frac{\sqrt{3}}{4\pi}\cos\left(\frac{4\pi x}{3}\right) + \dots \end{aligned}$$

2. Since the extended $\phi(x)$ and $\phi'(x)$ are piecewise continuous, the Fourier series converges to the average of the left & right limits at each point. Denote the sum of this series as $S(x)$, then

$$S(x) = \begin{cases} 0 & 0 \leq x < 1, \\ \frac{1}{2} & x = 1, \\ 1 & 1 < x \leq 3. \end{cases}$$

3. Since the extended $\phi(x)$ has finite L^2 norm, Parseval's equality holds, and its Fourier series converges in L^2 sense.

4. Take $x = 0$ in the series, we have

$$\begin{aligned} 0 = S(0) &= \frac{2}{3} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{3}\right) \\ &= \frac{2}{3} - \frac{\sqrt{3}}{\pi} K, \end{aligned}$$

where K is the sum we are trying to obtain

$$K = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \dots$$

Therefore,

$$\begin{aligned} K &= \frac{2}{3} \frac{\pi}{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{9} \pi. \end{aligned}$$

□

Problem 5. *(Page 134 135, Q7). Let*

$$\phi(x) = \begin{cases} -1 - x & \textit{for} \quad -1 < x < 0 \\ +1 - x & \textit{for} \quad 0 < x < 1. \end{cases}$$

1. *Find the full Fourier series of $\phi(x)$ in the interval $(-1, 1)$.*
2. *Find the first three nonzero terms explicitly.*
3. *Does it converge in the mean square sense?*
4. *Does it converge pointwise?*
5. *Does it converge uniformly to $\phi(x)$ in the interval $(-1, 1)$?*

Solution. It is noted that $\phi(-x) = -\phi(x)$, i.e., $\phi(x)$ is an odd function.

1. The full Fourier series of $\phi(x)$ collapses to the Fourier sine series

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x),$$

where

$$\begin{aligned}
A_m &= 2\int_0^1 \phi(x)\sin(m\pi x)\,dx \\
&= 2\int_0^1 (1-x)\sin(m\pi x)dx \\
&= -\frac{2}{m\pi}\int_0^1 (1-x)d\cos(m\pi x) \\
&= \frac{2}{m\pi} - \frac{2}{m\pi}\int_0^1 \cos(m\pi x)dx \\
&= \frac{2}{m\pi}.
\end{aligned}$$

Therefore,

$$\phi(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x).$$

2. The first three nonzero terms are $\frac{2}{\pi}\sin(\pi x)$, $\frac{1}{\pi}\sin(2\pi x)$ and $\frac{2}{3\pi}\sin(3\pi x)$.

3. Yes. Since $\phi(x)$ has finite L^2 norm, Parseval's equality holds.

4. Yes. (In $(-1,0)\cup(0,1)$).

5. No. Due to the jump in $\phi(x)$ at $x=0$, Gibbs phenomena appears and ruins uniform convergence.

   To better illustrate this, we use the complex Fourier series, which are mathematically equivalent to the real ones. Define the partial sum of the Fourier series as

$$\begin{aligned}
\phi_N(x) &= \sum_{n=-N}^{+N} U_n e^{\frac{i2\pi n}{2}x} \\
&= \sum_{n=-N}^{+N}\left[\frac{1}{2}\int_{-1}^{1}\phi(s)e^{-\frac{i2\pi n}{2}s}ds\right]e^{\frac{i2\pi n}{2}x}.
\end{aligned}$$

   Now we swap the order of the integration and the finite summation (this is ok since the integral converges for $\forall n$),

$$\begin{aligned}
\phi_N(x) &= \frac{1}{2}\int_{-1}^{1} \sum_{n=-N}^{+N} \phi(s) e^{\frac{i2\pi n}{2}(x-s)} ds \\
&= \frac{1}{2}\int_{-1}^{1} \phi(s) \left[\sum_{n=-N}^{+N} e^{\frac{i2\pi n}{2}(x-s)}\right] ds \\
&= \frac{1}{2}\int_{-1}^{1} \phi(s) \frac{e^{i\pi(x-s)(-N)} - e^{i\pi(x-s)(N+1)}}{1 - e^{i\pi(x-s)}} ds \\
&= \frac{1}{2}\int_{-1}^{1} \phi(s) \frac{e^{i\pi(x-s)(-N-0.5)} - e^{i\pi(x-s)(N+0.5)}}{e^{i\pi(x-s)(-0.5)} - e^{i\pi(x-s)(0.5)}} ds \\
&= \frac{1}{2}\int_{-1}^{1} \phi(s) \frac{\sin\left[\frac{\pi}{2}(x-s)(2N+1)\right]}{\sin\left[\frac{\pi}{2}(x-s)\right]} ds.
\end{aligned}$$

It is natural at this moment to define a kernel (Dirichlet kernel)

$$D_N(x) = \frac{\sin\left[(N+0.5)x\right]}{\sin(0.5x)},$$

such that we can rewrite the partial sum as

$$\phi_N(x) = \frac{1}{2}\int_{-1}^{1} \phi(s) D_N\left(\pi(x-s)\right) ds.$$

The Dirichlet kernel is even, 2π-periodic, and satisfy the following properties:

$$\begin{aligned}
\int_{-\pi}^{\pi} D_N(x) dx &= 2\pi, \\
D_N(0) &= 2N+1 \quad \text{(Peak value)}, \\
D_N\left(\pm\frac{2\pi}{2N+1}\right) &= 0 \quad \text{(First zeros)},
\end{aligned}$$

and the main lobe is defined as $\left(-\frac{2\pi}{2N+1}, \frac{2\pi}{2N+1}\right)$, over which the integral

$$\begin{aligned}
\int_{-\frac{2\pi}{2N+1}}^{\frac{2\pi}{2N+1}} D_N(x) dx &= 2\int_{0}^{\frac{2\pi}{2N+1}} D_N(x) dx \\
&= 2\int_{0}^{\frac{2\pi}{2N+1}} \frac{\sin\left[(N+0.5)x\right]}{\sin(0.5x)} dx \\
&= \frac{2}{N+0.5}\int_{0}^{\frac{2\pi}{2N+1}} \frac{\sin\left[(N+0.5)x\right]}{\sin\left((N+0.5)x\frac{0.5}{N+0.5}\right)} d[(N+0.5)x].
\end{aligned}$$

Let $y = (N + 0.5)x$ in the above integral, we have for large N,

$$\begin{aligned}\int_{-\frac{2\pi}{2N+1}}^{\frac{2\pi}{2N+1}} D_N(x)dx &= \frac{2}{N+0.5}\int_0^{\pi} \frac{\sin y}{\sin\left(\frac{1}{2N+1}y\right)}dy \\ &\geq \frac{2}{N+0.5}\int_0^{\pi} \frac{\sin y}{\frac{1}{2N+1}y}dy \\ &= 4\int_0^{\pi} \frac{\sin y}{y}dy \\ &= 4Si(\pi) \approx 7.0775 \approx 2\pi + 1.1246.\end{aligned}$$

This implies that as $N \to \infty$, the integral of $D_N(x)$ over an increasingly narrow strip near zero is non-vanishing.

Now we are ready to show the Gibbs phenomena. For simplicity, we define $f(x) = \phi(x) + x + 1$, and denote the partial sum of its Fourier series as $f_N(x)$.

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 2, & 0 < x < 1. \end{cases}$$

Now, consider $x_N = \frac{2}{2N+1}$,

$$\begin{aligned} f_N(x_N) &= \frac{1}{2}\int_{-1}^{1} f(s)D_N\left(\pi(s - x_N)\right)ds \\ &= \frac{1}{2}\int_0^1 2D_N\left(\pi(s - x_N)\right)ds \\ &= \frac{1}{\pi}\int_{-\pi x_N}^{\pi - \pi x_N} D_N\left(t\right)dt \\ &= \frac{1}{\pi}\int_{-\pi x_N}^{0} D_N\left(t\right)dt + \frac{1}{\pi}\int_0^{\pi} D_N\left(t\right)dt - \frac{1}{\pi}\int_{\pi - \pi x_N}^{\pi} D_N\left(t\right)dt. \end{aligned}$$

When N large, the first integral is approximated by $\frac{2Si(\pi)}{\pi} \approx 1.17898$, the second integral is equal to 1, and the third integral goes to zero. Therefore,

$$f_N(x_N) \approx 2.17898.$$

Consequently, $f_N(x)$ does not converge uniformly in any neighborhood of zero.

Note that by linearity, $P_N(x) = f_N(x) - \phi_N(x)$ is the partial sum of Fourier series for $P(x) = x + 1$. In a neighborhood of zero $(-r, r)$, $r < 1$, $P_N(x)$ converges uniformly, since it satisfies uniform two sided Lipschitz condition in $(-r, r)$. Then $\phi_N(x)$ must not converge uniformly.

$\square$