Homework #06 Answers and Hints
(MATH4052 Partial Differential Equations)

Xiaoyu Wei*

October 17, 2016

Problem 1. *(Page 45, Q2). Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l, 0 \le t < \infty\}$.*

1. *Let $M(T)$ = the maximum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?*
2. *Let $m(T)$ = the minimum of $u(x,t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?*

Solution. Denote $\Omega(T) = \{(x,t) | 0 \le x \le l, 0 \le t \le T\}$. Then we have

$$\Omega(T) \subset \Omega(T+h), \quad \forall h > 0.$$

The maximum (minimum) over a subset should be no more (less) than that over the whole; therefore, $M(T)$ should be an increasing function of T, and $m(T)$ a decreasing one. □

Problem 2. *(Page 46, Q4). Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1, 0 < t < \infty\}$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.*

1. *Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.*
2. *Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.*
3. *Use the energy method to show that $\int_0^1 u^2 dx$ is a strictly decreasing function of t.*

Solution. (Note: here we use the strong maximum principle without proving.)

1. By the (weak) maximum principle, we have

   $$0 \le u(x,t) \le 1, \quad \forall t \ge 0, 0 \le x \le 1.$$

   To show that the max/min values cannot be attained in the interior points, we have to use the strong (Hopf) maximum principle. Since the initial condition is not constant, we have

   $$0 < u(x,t) < 1, \quad \forall t > 0, 0 < x < 1.$$

*For any issues, email me at xweiaf@connect.ust.hk.

2. Let $v(x,t) = u(x,t) - u(1-x,t)$, then v solves the diffusion equation

$$v_t = v_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty,$$

subject to

$$\begin{aligned} v(0,t) = v(1,t) = 0, \\ v(x,0) = 0. \end{aligned}$$

Therefore, $v(x,t) = 0$, yielding

$$u(x,t) = u(1-x,t).$$

3. Let $E(t) = \int_0^1 u^2(x,t)dx$, then

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_0^1 2uu_t dx \\ &= \int_0^1 2uu_{xx} dx \\ &= 2uu_x|_0^1 - \int_0^1 2u_x u_x dx \\ &= -2\int_0^1 (u_x)^2 dx \\ &\leq 0, \end{aligned}$$

where the "=" can be attained if and only if $u(x,t)$ is a constant. We have shown previously that u is non-constant. Consequently, $E(t)$ is strictly decreasing.

□