Homework #05 Answers and Hints
(MATH4052 Partial Differential Equations)

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Problem 1. *(Page 89, Q2). Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature = 1. Suddenly both ends are plunged into a bath of temperature = 0. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, assume the infinite series expansion*

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right) \tag{1}$$

Solution. In idealized cases, we have

$$\begin{aligned} &\text{DE: } u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty),\\ &\text{BC: } u(0, t) = u(l, t) = 0 \quad (0 \le t < \infty),\\ &\text{IC: } u(x, 0) = 1 \quad (0 < x < l). \end{aligned}$$

It is a (homogeneous) Dirichlet problem. To solve it, we separate the variables $u(x, t) = T(t)X(x)$ and derive

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{constant}.$$

Therefore, $T(t)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T(t) = Ae^{-\lambda kt}$. For simplicity, we set $A = 1$, so that the initial condition is directly satisfied by $X(x)$. Furthermore,

$$-X'' = \lambda X \quad \text{in } 0 < x < l \quad \text{with } X(0) = X(l) = 0.$$

Using the boundary conditions, we have $\lambda = \frac{n^2\pi^2}{l^2}$, and

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l},$$

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where the coefficients are given by the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}.$$

Applying the initial condition, we have

$$\begin{aligned} A_n &= \frac{\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx}{\int_0^l \sin^2\frac{n\pi x}{l}dx}. \\ &= \frac{2\int_0^l \sin\frac{n\pi x}{l}dx}{\int_0^l 1-\cos\frac{2n\pi x}{l}dx}. \\ &= \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}dx. \\ &= \frac{2}{l}\frac{l}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]\Big|_0^l \\ &= \frac{2}{n\pi}\left[1-\cos(n\pi)\right], \end{aligned}$$

which is 0 for even n and $\frac{4}{n\pi}$ for odd n. Alternatively this can be obtained using the given formula.

In summary, we have

$$u(x,t) = \sum_{n \text{ odd}} \frac{4}{n\pi} e^{-\frac{n^2\pi^2 kt}{l^2}} \sin\frac{n\pi x}{l}.$$

□

Problem 2. *(Page 89, Q4). Consider waves in a resistant medium that satisfy the problem*

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - r u_t \quad \textit{for} \quad 0 < x < l, && (2) \\ u &= 0 \quad \textit{at both ends}, && (3) \\ u(x,0) &= \phi(x), && (4) \\ u_t(x,0) &= \psi(x), && (5) \end{aligned}$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

Solution. The equation is very similar to the wave equation, so we can try whether separation of variables still work. Let $u(x,t) = T(t)X(x)$, we have from the same argument

$$\frac{T''}{c^2T} + \frac{r}{c^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{constant}.$$

For $X(x)$, we have

$$-X'' = \lambda X \quad \text{in } 0 < x < l \quad \text{with } X(0) = X(l) = 0.$$

Just like before, from the boundary conditions, we have $\lambda_n = \frac{n^2\pi^2}{l^2}$ so that nontrivial solution exists.

Let $X_n(x)$ solve the ODE $-X'' = \lambda_n X$ and $T_n(t)$ solve the ODE $T'' + rT' + c^2\lambda_n T = 0$, then

$$X_n(x) = \sin\frac{n\pi x}{l},$$

and

$$T_n(t) = A_n e^{s_n^{(1)}t} + B_n e^{s_n^{(2)}t},$$

where $s_n^{(1)} \neq s_n^{(2)} \in \mathbb{C}$ are two roots of the quadratic equation $s^2 + rs + c^2\lambda_n = 0$. Note that if $s_n^{(1)} = s_n^{(2)} = s_n$, that is, when $n = \frac{lr}{2\pi c}$,

$$T_n(t) = A_n e^{s_n t} + B_n t e^{s_n t}.$$

Since $0 < r < 2\pi c/l$, $\frac{lr}{2\pi c} \notin \mathbb{N}$, and the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty}\left[A_n e^{s_n^{(1)}t} + B_n e^{s_n^{(2)}t}\right]\sin\frac{n\pi x}{l},$$

where the coefficients are given by the initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty}[A_n + B_n]\sin\frac{n\pi x}{l},$$

and

$$u_t(x,0) = \psi(x) = \sum_{n=1}^{\infty}\left[A_n s_n^{(1)} + B_n s_n^{(2)}\right]\sin\frac{n\pi x}{l}.$$

Given that $\phi(x) = \sum_{n=1}^{\infty} P_n \sin\frac{n\pi x}{l}$ and $\psi(x) = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{l}$, the coefficients solve a linear system

$$\begin{bmatrix} I_n & I_n \\ S_1 & S_2 \end{bmatrix}\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} P_n \\ Q_n \end{bmatrix},$$

where S_1, S_2 are diagonal matrices with the roots above. Since $s_n^{(1)} \neq s_n^{(2)}$, the coefficient matrix is of full rank and the system admits unique solution.

Remark 1. *On the other hand, if $m := \frac{lr}{2\pi c} \in \mathbb{N}$,*

$$u(x,t) = \left[A_m e^{s_m t} + B_m t e^{s_m t}\right]\sin\frac{m\pi x}{l} + \sum_{n\neq m}\left[A_n e^{s_n^{(1)}t} + B_n e^{s_n^{(2)}t}\right]\sin\frac{n\pi x}{l},$$

where the coefficients are given by the initial condition

$$u(x,0) = \phi(x) = A_m \sin\frac{m\pi x}{l} + \sum_{n\neq m}[A_n + B_n]\sin\frac{n\pi x}{l},$$

and

$$u_t(x,0) = \psi(x) = [s_m A_m + B_m] \sin \frac{m\pi x}{l} + \sum_{n \neq m} \left[A_n s_n^{(1)} + B_n s_n^{(2)} \right] \sin \frac{n\pi x}{l}.$$

Now, if $s_m \neq 1$, the system matrix is also of full rank, and the solution is unique. Actually, since

$$s_m = \frac{-r}{2} < 0,$$

this is the only possible case.

□

Problem 3. *(Page 92, Q2). Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,t) = 0, u(l,t) = 0$ (Neumann at the left, Dirichlet at the right).*

1. *Show that the eigenfunctions are $\cos\left[\left(n + \frac{1}{2}\right)\pi x/l\right]$.*
2. *Write the series expansion for a solution $u(x,t)$.*

Solution. Separation of variable $u(x,t) = X(x)T(t)$ yields

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{constant}.$$

In search for positive eigenvalues $\lambda = \beta^2 > 0$, we have

$$X(x) = C \cos \beta x + D \sin \beta x,$$
$$X'(x) = -C\beta \sin \beta x + D \cos \beta x.$$

To satisfy the boundary conditions, we have

$$0 = X'(0) = D,$$
$$0 = X(l) = C \cos \beta l + D \sin \beta l.$$

Since C, D cannot be all zeros, it must be that $\cos \beta l = 0$, i.e.

$$\beta l = \frac{\pi}{2} + m\pi, \quad m \in \mathbb{Z},$$

giving the eigenvalues

$$\lambda_n = \left(\frac{2n-1}{2} \frac{\pi}{l} \right)^2.$$

And the eigenfunctions (taking $C = 1$ for simplicity)

$$X_n(x) = \cos \left[\left(n + \frac{1}{2} \right) \frac{\pi x}{l} \right].$$

The corresponding $T_n(t)$ is

$$T_n(t) = e^{-k\lambda_n t} = e^{-k\left(\frac{2n-1}{2}\frac{\pi}{l}\right)^2 t},$$

and thus the series expansion for the solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n T_n(t) X_n(x),$$

where the coefficients A_n are given by series expansion of initial condition

$$\begin{aligned} u(x,0) = \phi(x) &= \sum_{n=1}^{\infty} A_n X_n(x) \\ &= \sum_{n=1}^{\infty} A_n \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right]. \end{aligned}$$

□

Problem 4. *(Page 92, Q3). Solve the Schrdinger equation* $u_t = iku_{xx}$ *for real* k *in the interval* $0 < x < l$ *with the boundary conditions* $u_x(0,t) = 0, u(l,t) = 0$.

Solution. Separation of variables $u(x,t) = X(x)T(t)$ leads to the equation

$$\frac{T'}{ikT} = \frac{X''}{X} = -\lambda = \text{constant},$$

so that $T(t) = e^{-ik\lambda t}$ and (using results from the previous problem) the eigenvalues are

$$\lambda_n = \left(\frac{2n-1}{2}\frac{\pi}{l}\right)^2,$$

with eigenfunctions

$$X_n(x) = \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right].$$

Then the solution can be written as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n T_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} A_n e^{-ik\lambda t} \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right] \\ &= \sum_{n=1}^{\infty} A_n \left\{\cos\left[kt\left(\frac{2n-1}{2}\frac{\pi}{l}\right)^2\right] - \sin\left[kt\left(\frac{2n-1}{2}\frac{\pi}{l}\right)^2\right]\right\} \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right], \end{aligned}$$

with the coefficients given by series expansion of initial conditions

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} A_n \cos\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right].$$

□