Homework #03 Answers and Hints
(MATH4052 Partial Differential Equations)

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Problem 1. *(Page 38, Q1). Solve* $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin x$.

Solution. Here we review the derivation of the solution formula. Assuming all functions hereafter are sufficiently smooth, the equation can be rewritten as

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x) u = 0, \tag{1}$$

from which we can solve

$$(\partial_t + c\partial_x) u = h(x + ct). \tag{2}$$

One solution to Equation (2) is

$$u(x,t) = f(x+ct), \quad f(s) = \frac{1}{2c}\int h(s)ds. \tag{3}$$

Then the general solution becomes

$$u(x,t) = f(x+ct) + g(x-ct), \quad f, g \in \mathcal{C}^2. \tag{4}$$

Now plug in the initial conditions

$$u(x,0) = f(x) + g(x) = e^x, \tag{5}$$
$$u_t(x,0) = cf'(x) - cg'(x) = \sin x, \tag{6}$$

which solves

$$f'(x) = \frac{1}{2}\left(e^x + \frac{1}{c}\sin x\right), \tag{7}$$
$$g'(x) = \frac{1}{2}\left(e^x - \frac{1}{c}\sin x\right). \tag{8}$$

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Integrate the derivatives to get (noticing that f and g adds up to e^x)

$$f(x) = \frac{1}{2}\left(e^x - \frac{1}{c}\cos x\right) + C, \tag{9}$$

$$g(x) = \frac{1}{2}\left(e^x + \frac{1}{c}\cos x\right) - C. \tag{10}$$

Therefore,

$$u(x,t) = \frac{1}{2}\left[e^{x+ct} - \frac{1}{c}\cos(x+ct)\right] + \frac{1}{2}\left[e^{x-ct} + \frac{1}{c}\cos(x-ct)\right] \tag{11}$$

$$= \frac{1}{2}\left[e^{x+ct} + e^{x-ct}\right] + \frac{1}{2c}\left[cos(x-ct) - cos(x+ct)\right]. \tag{12}$$

The solution formula is the celebrated d'Alembert formula. □

Problem 2. *(Page 38, Q2). Solve* $u_{tt} = c^2 u_{xx}$, $u(x,0) = \log(1+x^2)$, $u_t(x,0) = 4+x$.

Solution. Directly apply d'Alembert formula, and we have

$$u(x,t) = \frac{1}{2}\left\{\log\left[1+(x+ct)^2\right] + \log\left[1+(x-ct)^2\right]\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)ds \tag{13}$$

$$= \frac{1}{2}\log\left\{\left[1+(x+ct)^2\right]\left[1+(x-ct)^2\right]\right\} + \frac{1}{2c}\left\{\left[4(x+ct) + \frac{1}{2}(x+ct)^2\right] - \left[4(x-ct) + \frac{1}{2}(x-ct)^2\right]\right\} \tag{14}$$

$$= \frac{1}{2}\log\left\{\left[1+(x+ct)^2\right]\left[1+(x-ct)^2\right]\right\} + \frac{1}{2c}\left(8ct + 2xct\right). \tag{15}$$

□

Problem 3. *(Page 38, Q9). Solve* $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. *(Hint: Factor the operator as we did for the wave equation.)*

Solution. The equation is of hyperbolic type, thus it can be rewritten as

$$\left(\partial_x - 4\partial_t\right)\left(\partial_x + \partial_t\right)u = 0, \tag{16}$$

or equivalently,

$$\left(\partial_t - \frac{1}{4}\partial_x\right)\left(\partial_t + \partial_x\right)u = 0, \tag{17}$$

Then by solving the outer transport equation, we have

$$\left(\partial_t + \partial_x\right)u = h\left(x + \frac{1}{4}t\right). \tag{18}$$

Noticing a particular solution to be

$$u(x,t) = f\left(x + \frac{1}{4}t\right), \quad f(s) = \frac{1}{1+\frac{1}{4}} \int h(s)ds = \frac{4}{5} \int h(s)ds, \tag{19}$$

we can write the general solution to be

$$u(x,t) = f\left(x + \frac{1}{4}t\right) + g(x-t). \tag{20}$$

Now, plug in the initial conditions to get

$$u(x,0) = f(x) + g(x) = x^2, \tag{21}$$

$$u_t(x,0) = \frac{1}{4}f'(x) - g'(x) = e^x. \tag{22}$$

Differentiate the first equation above and solve with the second one to give

$$f'(x) = \frac{8}{5}x + \frac{4}{5}e^x, \tag{23}$$

$$g'(x) = \frac{2}{5}x - \frac{4}{5}e^x. \tag{24}$$

Integrating the derivatives yield

$$f(x) = \frac{4}{5}x^2 + \frac{4}{5}e^x + C, \tag{25}$$

$$g(x) = \frac{1}{5}x^2 - \frac{4}{5}e^x - C. \tag{26}$$

Therefore, the solution is

$$u(x,t) = \left[\frac{4}{5}\left(x + \frac{1}{4}t\right)^2 + \frac{4}{5}e^{x+\frac{1}{4}t}\right] + \left[\frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t}\right]. \tag{27}$$

□

Problem 4. *(Page 41, Q4). If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity*

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h) \tag{28}$$

for all $x, t, h,$ and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.

Solution. Since u is a solution to the wave equation, we have

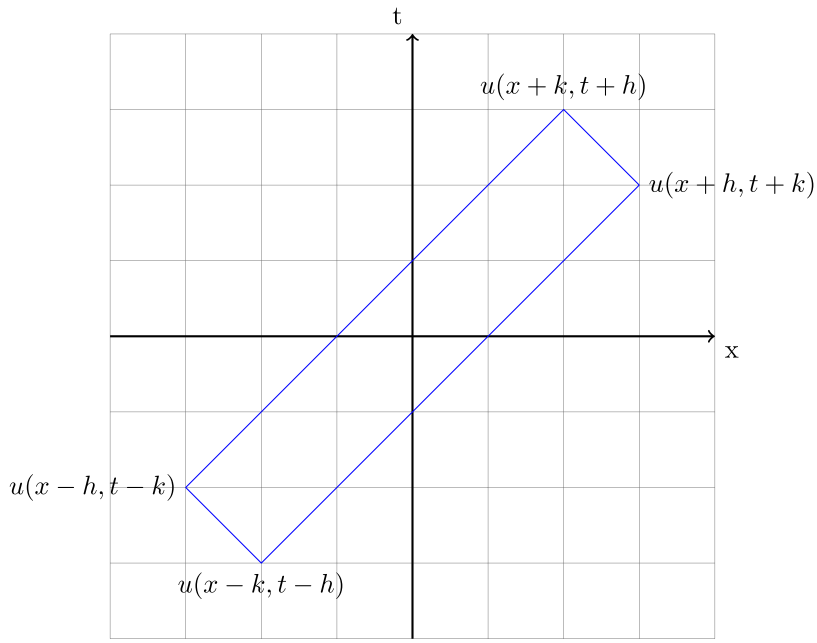
$$u(x,t) = f(x+t) + g(x-t). \tag{29}$$

Then, denoting the left hand side of the identity as L, and the right hand side as R ,

$$\begin{aligned} L &= [f(x+t+h+k) + g(x-t+h-k)]+ \\ &\quad [f(x+t-h-k) + g(x-t-h+k)] \quad (30) \\ &= [f(x+t+h+k) + g(x-t-h+k)]+ \\ &\quad [f(x+t-h-k) + g(x-t+h-k)] \quad (31) \\ &= u(x+k, t+h) + u(x-k, t-h) = R \quad (32) \end{aligned}$$

A plot illustrating the quadrilateral Q is shown in Figure 1. It is seen that Q is a rectangle.

Figure 1: Quadrilateral Q.



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