Homework #02 Answers and Hints (MATH4052 Partial Differential Equations)

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Problem 1. *(Page 27, Q3). Solve the boundary problem $u'' = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the $+$ and $-$ cases separately. What is special about the case $k = 2$?*

Solution. Integrating the equation twice, we have

$$u'(x) = C, \tag{1}$$
$$u(x) = Cx + D, \tag{2}$$

where C, D are constants.

First we consider the $+$ case. In this case the two boundary conditions become

$$C + kD = 0 \tag{3}$$
$$C + k(C + D) = 0, \tag{4}$$

solving $C = 0, D = 0$ if $k \neq 0$. If $k = 0$, we still have $C = 0$, but D can take any value. Therefore, the solution to the $+$ case is:

$$u(x) = \begin{cases} 0, & if \quad k \neq 0, \\ D, & D \in \mathbb{R}, \quad if \quad k = 0. \end{cases} \tag{5}$$

On the other hand, for the $-$ case, the two boundary conditions become

$$C + kD = 0, \tag{6}$$
$$C - k(C + D) = 0, \tag{7}$$

solving

$$u(x) = \begin{cases} 0, & if \quad k \neq 0, 2, \\ D, & if \quad k = 0, \\ -2Dx + D, & if \quad k = 2, \end{cases} \tag{8}$$

where $D \in \mathbb{R}$. So only when $k = 2$ does the problem have nontrivial solution.

□

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Problem 2. *(Page 31, Q1). What is the type of each of the following equations?*

1. $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$
2. $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

Solution. Denote the (second order) principle part as $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}$.

For the first equation, the determinant $\mathscr{D} = a_{12}^2 - a_{11}a_{22} = (-2)^2 - 1 \times 1 = 3$. So it is hyperbolic.

Similarly, for the second equation, $\mathscr{D} = 3^2 - 9 \times 1 = 0$; therefore it is parabolic. □

Problem 3. *(Page 32, Q6). Consider the equation* $3u_y + u_{xy} = 0$.

1. *What is its type?*
2. *Find the general solution. (Hint: Substitute* $v = u_y$*.)*
3. *With the auxiliary conditions* $u(x, 0) = e^{-3x}$ *and* $u_y(x, 0) = 0$*, does a solution exist? Is it unique?*

Solution. From the equation,

1. $\mathscr{D} = \left(\frac{1}{2}\right)^2 - 0 > 0$. So it is hyperbolic.
2. Substituting $v = u_y$, the equation becomes

$$3v + v_x = 0. \tag{9}$$

Thus

$$v(x, y) = v(0, y)e^{-3x}. \tag{10}$$

Here we implicitly change the order of differentiation. The price we have to pay is that our solution should sit in $\mathcal{C}^2(\mathbb{R})$.

Substitute back and we have

$$u_y = C(y)e^{-3x}, \quad C(y) \in \mathcal{C}(\mathbb{R}), \tag{11}$$

solving

$$u(x, y) = e^{-3x}\left[\int C(y)dy\right] + D(x). \tag{12}$$

Therefore, the general solution is

$$u(x, y) = e^{-3x}f(y) + D(x), \quad f(y), D(x) \in \mathcal{C}^2(\mathbb{R}). \tag{13}$$

3. To have $u(x, 0) = e^{-3x}$, we can let

$$f(0) = 1, \tag{14}$$

$$D(x) = 0, \tag{15}$$

then from $u_y(x, 0) = 0$, we have

$$f'(y) = 0. \tag{16}$$

Obviously such a solution exists, for example,

$$u(x, y) = e^{-3x}. \tag{17}$$

However, the solution above is not the only one. Another example is

$$u(x, y) = e^{-3x} \left(1 + y^2\right). \tag{18}$$

Therefore, solution exists, but is not unique.

□