Homework #01 Answers and Hints
(MATH4052 Partial Differential Equations)

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Problem 1. *(Page 5, Q2(a)(b)). Which of the following operators are linear?*

1. $\mathscr{L}u = u_x + xu_y$
2. $\mathscr{L}u = u_x + uu_y$

Solution. Checking with the definition of linear operators on Page. 2:

1. $\mathscr{L}(u+v) = (u_x + v_x) + x(u_y + v_y) = \mathscr{L}u + \mathscr{L}v$.
   $\mathscr{L}(cu) = cu_x + cxu_y = c\mathscr{L}u$.
2. $\mathscr{L}(u+v) = (u_x + v_x) + (u+v)(u_y + v_y) \neq \mathscr{L}u + \mathscr{L}v$.

Therefore, the first one is linear, while the second one is not. □

Problem 2. *(Page 5, Q3(c)(d)). For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.*

1. $u_t - u_{xxt} + uu_x = 0$
2. $u_{tt} - u_{xx} + x^2 = 0$

Solution. For the first equation, $\mathscr{L}u = u_t - u_{xxt} + uu_x$ is a third order nonlinear operator since $\mathscr{L}(u+v) \neq \mathscr{L}u + \mathscr{L}v$; therefore, it is a 3rd-order nonlinear equation.

For the second one, $\mathscr{L}u = u_{tt} - u_{xx}$ is a second order linear operator since $\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v$ and $\mathscr{L}(cu) = c\mathscr{L}u$. Besides, the equation is in the form of $\mathscr{L}u = g(x)$ where $g(x) = x^2$; therefore, this equation is a 2nd-order linear inhomogeneous equation.

□

Problem 3. *(Page 9, Q1). Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.*

Solution. Denote space-time gradient $\nabla = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)^T$, and the equation can be rewritten as

$$\mathbf{v} \cdot \nabla u = 0,$$

where $\mathbf{v} = (2, 3)^T$ is a constant vector. This implies that $u(t, x)$ is constant along $\mathbf{v}$. In other words, the value of u at point $(t, x)^T$ equals to that at $(t, x)^T + \lambda \mathbf{v}$ for arbitrary λ.

Here we use the geometric method since the characteristics are easy to find.

Now, take $\lambda = -t/2$ and we have

$$u(t, x) = u\left(0, x - \frac{3}{2}t\right) = \sin\left(x - \frac{3}{2}t\right).$$

□

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