MATH 2352 Solution Sheet 11

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[Problems] 8.1: 12(a,c), 23; 8.2: 4(b), 22; 8.3: 1(b).

8.1 - 12. For the initial value problem

$$\begin{aligned} y' &= (y^2+2ty)/(3+t^2), \\ y(0) &= 0.5, \end{aligned}$$

find the approximate values of the solution at $t=0.5, 1.0, 1.5$ and 2.0.

(a) Use the Euler method with $h=0.025$.

(c) Use the backward Euler method with $h=0.025$.

Solution. For error analysis, we give the analytic solution here.

$$\begin{aligned} y' &= (y^2+2ty)/(3+t^2) \\ (t^2+3)y' - 2ty - y^2 &= 0 \\ \frac{y'}{t^2+3} - \frac{2t}{t^2+3}\frac{y}{t^2+3} - \left(\frac{y}{t^2+3}\right)^2 &= 0 \\ \left(\frac{y}{t^2+3}\right)' &= \left(\frac{y}{t^2+3}\right)^2. \end{aligned}$$

Let $\frac{y}{t^2+3} = u$,

$$\begin{aligned} \frac{\mathrm{du}}{u^2} &= \mathrm{dt} \\ -\frac{1}{u} &= t + C \\ u &= \frac{-1}{t+C}. \end{aligned}$$

Therefore,

$$y(t) = \frac{-(t^2+3)}{t+C}.$$

By $y(0) = 0.5$,

$$y(t) = \frac{-(t^2+3)}{t-6}.$$

8.1 - 23. In this problem we discuss the global truncation error associated with the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. Assuming that the functions f and f_y are continuous in a closed, bounded region R of the ty-plane that includes the point (t_0, y_0), it can be shown that there exists a constant L such that $|f(t,y) - f(t,\tilde{y})| < L|y - \tilde{y}|$, where (t,y) and $(t,\tilde{y})$ are any two points in R with the same t coordinate. Further, we assume that f_t is continuous, so the solution ϕ has a continuous derivative.

(a) Using

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)\,h + \frac{1}{2}\phi''(\bar{t}_n)\,h^2,$$

show that

$$\begin{aligned} |E_{n+1}| &\leqslant |E_n| + h\,|f[t_n, \phi(t_n)] - f(t_n, y_n)| + \frac{1}{2}h^2|\phi''(\bar{t}_n)| \qquad (1) \\ &\leqslant \alpha|E_n| + \beta h^2, \end{aligned}$$

where $\alpha = 1 + hL$ and $\beta = \max|\phi''(t)|/2$ on $t_0 \leqslant t \leqslant t_n$.

(b) Assume that if $E_0 = 0$ and if $|E_n|$ satisfies the Eqn (1), then $|E_n| \leqslant \beta h^2(\alpha^n - 1)/(\alpha - 1)$ for $\alpha \neq 1$. Use this result to show that

$$|E_n| \leqslant \frac{(1 + hL)^n - 1}{L}\beta h. \qquad (2)$$

Eqn (2) gives a bound for $|E_n|$ in terms of h, L, n and β. Notice that for a fixed h, this error bound increases with increasing n; that is, the error bound increases with distance from the starting point t_0.

(c) Show that $(1 + hL)^n \leqslant e^{nhL}$; hence

$$|E_n| \leqslant \frac{e^{nhL} - 1}{L}\beta h. \qquad (3)$$

If we select an ending point T greater than t_0 and then choose the step size h so that n steps are required to traverse the interval $[t_0, T]$, then $nh = T - t_0$, and

$$|E_n| \leqslant \frac{e^{(T - t_0)L} - 1}{L}\beta h = Kh. \qquad (4)$$

Note that K depends on the length $T - t_0$ of the interval and on the constants L and β that are determined from the function f.

Solution.

(a) By evaluating truncation error,

$$\begin{aligned} |E_{n+1}| &\leqslant |E_n| + h\,|f[t_n, \phi(t_n)] - f(t_n, y_n)| + \frac{1}{2}h^2|\phi''(\bar{t}_n)| \\ &\leqslant |E_n| + hL|\phi(t_n) - y_n| + \frac{1}{2}h^2\beta \\ &= \alpha|E_n| + \beta h^2. \end{aligned}$$

(b) Since

$$\begin{aligned}
|E_n| &\leqslant \alpha|E_{n-1}|+\beta h^2\\
&\leqslant \alpha(\alpha|E_{n-2}|+\beta h^2)+\beta h^2\\
&\leqslant \dots\\
&\leqslant \alpha^n|E_0|+\sum_{k=0}^{n-1}\alpha^k\beta h^2\\
&= \sum_{k=0}^{n-1}\alpha^k\beta h^2\\
&= \frac{1-\alpha^n}{1-\alpha}\beta h^2\\
&= \frac{(1+hL)^n-1}{L}\,\beta h.
\end{aligned}$$

(c) The conclusion is natural.

8.2 - 4. For the initial value problem

$$\begin{aligned}
y' &= 2t+e^{-ty},\\
y(0) &= 1,
\end{aligned}$$

find the approximate values of the solution at $t=0.1, 0.2, 0.3$ and 0.4. Compare the results with those obtained by the Euler method and with the exact solution (if available).

(b) Use the Euler method with $h=0.025$.

8.2 - 22. The **modified Euler formula** for the initial value problem $y'=f(t,y)$, $y(t_0)=y_0$ is given by

$$y_{n+1} = y_n+hf\Big[t_n+\frac{1}{2}h, y_n+\frac{1}{2}h\,f(t_n,y_n)\Big].$$

Show that the local truncation error in the modified Euler formula is proportional to h^3.

Solution. Let

$$\begin{aligned}
K_1 &= h\,f(t_n,y_n),\\
K_2 &= hf\Big[t_n+\frac{1}{2}h,\, y_n+\frac{1}{2}h\,f(t_n,y_n)\Big],
\end{aligned}$$

then the approximation is

$$y_{n+1} \approx y_n+K_2.$$

The Taylor expansion of f yields

$$\begin{aligned}
K_2 &= h\Big[f(t_n,y_n)+\frac{h}{2}\frac{\partial}{\partial t}f(t_n,y_n)+\frac{K_1}{2}\frac{\partial}{\partial y}f(t_n,y_n)+O(h^2,K_1^2)\Big]\\
&= hf(t_n,y_n)+\frac{h^2}{2}\Big[\frac{\partial}{\partial t}f(t_n,y_n)+f(t_n,y_n)\frac{\partial}{\partial y}f(t_n,y_n)\Big]+O(h^3).
\end{aligned}$$

On the other hand, Taylor expansion of y yields

$$y(t_n+h) = y(t_n)+hy'(t_n)+\frac{h^2}{2}y''(t_n)+O(h^3).$$

Substitude

$$\begin{aligned} y'(t_n) &= f(t_n, y(t_n)) \\ y''(t_n) &= \frac{d}{\mathrm{dt}}f(t_n, y(t_n)) \\ &= \frac{\partial}{\partial t}f(t_n, y_n)+(t_n, y_n)\frac{\partial}{\partial y}f(t_n, y_n), \end{aligned}$$

to get

$$\begin{aligned} y(t_n+h) &= y(t_n)+hf(t_n, y(t_n))+\frac{h^2}{2}\left[\frac{\partial}{\partial t}f(t_n, y_n)+(t_n, y_n)\frac{\partial}{\partial y}f(t_n, y_n)\right]+O(h^3) \\ &= y(t_n)+K_2+O(h^3). \end{aligned}$$

Therefore, the local truncation error

$$\tau_n = O(h^3).$$

8.3 - 1. For the initial value problem

$$\begin{aligned} y' &= 2+t-y, \\ y(0) &= 1, \end{aligned}$$

find the approximate values of the solution at $t=0.1, 0.2, 0.3$ and 0.4. Compare the results with those obtained by the numerical method and with the exact solution (if available).

(b) Use the Runge-Kutta method with $h=0.05$.

Solution. Exact solution is $y(t)=Ce^{-t}+t+1$, where $C=0$ for $y(0)=1$.