MATH 2352 Solution Sheet 10

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[Problems] 7.7: 6, 12, 13; 7.8: 7, 18, 22 7.9: 1, 3.

7.7 - 6. For

$$x' = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix} x,$$

(a) Find a fundamental matrix for the given system of equations.

(b) Also find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$.

Solution.

(a) The general solution is

$$x(t) = c_1 \begin{bmatrix} e^{-t}\cos(2t) \\ -2e^{-t}\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin(2t) \\ 2e^{-t}\cos(2t) \end{bmatrix}.$$

Then a fundamental matrix can be formed using any two linearly independent solutions, for example,

$$\Psi(t) = \begin{bmatrix} e^{-t}\sin(2t) & e^{-t}\cos(2t) \\ 2e^{-t}\cos(2t) & -2e^{-t}\sin(2t) \end{bmatrix}.$$

(b) The fundamental matrix can be normalized using properly chosen c_1, c_2,

$$\Phi(t) = \begin{bmatrix} e^{-t}\cos(2t) & \frac{1}{2}e^{-t}\sin(2t) \\ -2e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix},$$

which satisfies $\Phi(0) = I$.

7.7 - 12. Solve the initial value problem

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x,$$

$$x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

by using the fundamental matrix $\Phi(t)$ for the following system of equations:

$$x' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} x.$$

Solution. We intend to use the relation $x(t) = \Phi(t)\, x(0)$.

To solve for $\Phi(t)$, we directly compute

$$\Phi(t) \;=\; e^{A t},$$

where

$$A \;=\; \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$

A can be decomposed as Jordan canonical form

$$\begin{aligned} A &= S J S^{-1} \\ &= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \end{aligned}$$

which can be obtained from the transposed matrix.

Then

$$\begin{aligned} e^{A t} &= S e^{J t} S^{-1} \\ &= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

To compute $x(t) = \Phi(t)\, x(0)$, we have

$$\begin{aligned} x(t) &= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}\begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{11}{2} e^{-t} \\ -\frac{5}{2} e^{t} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{11}{2} e^{-t} + \frac{15}{2} e^{t} \\ \frac{11}{2} e^{-t} - \frac{5}{2} e^{t} \end{pmatrix}. \end{aligned}$$

7.7 - 13. Show that $\Phi(t) = \Psi(t)\, \Psi^{-1}(t_0)$, where $\Phi(t)$ and $\Psi(t)$ are as defined in this section (or as in the slides).

Solution. It is equivalent to show

$$\Phi(t) \Psi(t_0) \;=\; \Psi(t).$$

Consider each column of the equaion above, it means that the solution forming the fundamental matrix is linear combinition of columns of $\Phi(t)$, with coefficients as initial values. This is exactly the definition of $\Phi(t)$.

7.8 - 7. Consider the initial value problem

$$\begin{aligned} x' &= \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x, \\ x(0) &= \begin{pmatrix} 4 \\ 2 \end{pmatrix}. \end{aligned}$$

(a) Find the solution.

(b) Draw the trajectory of the solution in the $x_1 x_2$-plane, and also draw the graph of x_1 versus t.

Solution.

(a) Same as the previous problem,

$$\begin{aligned} x(t) &= \Phi(t)\, x(0) \\ &= e^{At} x(0) \qquad \text{(Jordan canonical form)} \\ &= S e^{Jt} S^{-1} x(0) \\ &= \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-3t} & te^{t} \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-3t} & te^{t} \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 2 \\ 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2e^{-3t} + 8te^{t} \\ 8e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} 4e^{-3t} + 8te^{t} \\ 2e^{-3t} + 8te^{t} \end{pmatrix}. \end{aligned}$$

(b) As shown below:

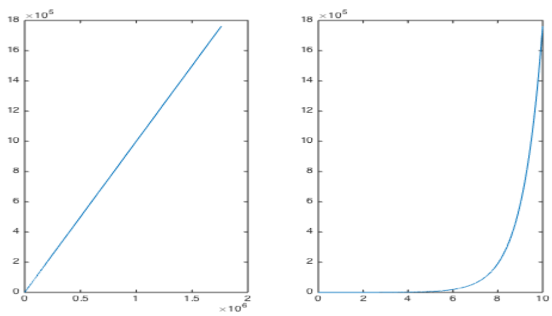

Figure 1. Trajector (left) and $x_1(t)$ (right)

7.8 - 18. Consider the system

$$x' = Ax = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} x. \tag{1}$$

(a) Show that $r=2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix A and that there is only one corresponding eigenvector, namely,

$$\xi^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $x^{(1)}(t)$ of the system (1). There is no other solution of the purely exponential form $x=\xi e^{rt}$.

(c) To find a second solution, assume that $x=\xi te^{2t}+\eta e^{2t}$. Show that ξ and η satisfy the equations

$$\begin{aligned}(A-2I)\,\xi &= 0,\\ (A-2I)\,\eta &= \xi.\end{aligned}$$

Since ξ has already been found in part (a), solve the second equation for η. Neglect the multiple of $\xi^{(1)}$ that appears in η, since it leads only to a multiple of the first solution $x^{(1)}$. Then write down a second solution $x^{(2)}(t)$ of the system (1).

(d) To find a third solution, assumen that $x=\xi\,(t^2/2)\,e^{2t}+\eta\,te^{2t}+\zeta\,e^{2t}$. Show that ξ, η and ζ satisfy the equations

$$\begin{aligned}(A-2I)\,\xi &= 0,\\ (A-2I)\,\eta &= \xi,\\ (A-2I)\,\zeta &= \eta.\end{aligned}$$

The first two equations are the same as in part (c), so solve the third equation for ζ, again neglecting the multiple of $\xi^{(1)}$ that appears. Then write down a third solution $x^{(3)}(t)$ of the system (1).

(e) Write down a fundamental matrix $\Psi(t)$ for the system (1).

(f) Form a matrix T with the eigenvector $\xi^{(1)}$ in the first column and the generalized eigenvector η and ζ in the second and third solumns. Then find T^{-1} and form the product $J=T^{-1}AT$. The matrix J is the Jordan form of A.

Solution.

The solution procedure is to evaluate the Jordan canonical form and solve the ODE in parallel. Following the instructions, we can get

$$A = \begin{pmatrix} 0 & -1 & -2 \\ -1 & -1 & -3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -3 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix},$$

and

$$x(t) = c_1\begin{pmatrix} -2e^{2t}(t-1) \\ -e^{2t}(t-4)t \\ e^{2t}(t-6)t \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t}t \\ e^{2t}[(t-2)t+2] \\ -e^{2t}(t-4)t \end{pmatrix} + c_3\begin{pmatrix} 2e^{2t}t \\ e^{2t}(t-2)t \\ -e^{2t}[(t-4)t-2] \end{pmatrix}.$$

7.8 - 22. Let

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find J^2, J^3, and J^4.

(b) Use an inductive argument to show that

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & [n(n-1)/2]\,\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine e^{Jt}.

(d) Note that if you choose $\lambda = 2$, then the matrix J in this problem is the same as the matrix J in Problem 18(f), form the product $T e^{Jt}$ with $\lambda = 2$. The resulting matrix is the same as the fundamental matrix $\Psi(t)$ in Problem 18(e).

Solution. The solution is straight foreward following the instructions, and to determine e^{Jt}, consider its (formal) Taylor expansion.

$$e^{Jt} = I + Jt + \frac{1}{2}J^2 t^2 + \ldots$$

7.9 - 1. Find the general solution of the given system of equations.

$$x' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

Solution. Solve the homogeneous equaion first, then find a special solution to the inhomogeneous equaion.

See the solution to the next problem for step-by-step guidlines.

7.9 - 3. Find the general solution of the given system of equations.

$$x' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} x + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

Solution.

1. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}.$$

We have

$$\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i, \end{aligned}$$

and

$$v_1 = \begin{pmatrix} -\frac{2}{5} - \frac{i}{5} \\ 1 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} -\frac{2}{5} + \frac{i}{5} \\ 1 \end{pmatrix},$$

2. Compute the solution to the homogeneous equation

$$\begin{aligned} x_1 &= e^{\lambda_1 t} v_1 \\ &= \begin{pmatrix} -\frac{2}{5} - \frac{i}{5} \\ 1 \end{pmatrix} e^{i t} \\ x_2 &= e^{\lambda_2 t} v_2 \\ &= \begin{pmatrix} -\frac{2}{5} + \frac{i}{5} \\ 1 \end{pmatrix} e^{-i t}. \end{aligned}$$

And let $X(t) = [\operatorname{Re}(x_1), \operatorname{Im}(x_1)]$ be a (real) fundamental matrix.

$$X(t) = \begin{pmatrix} -\frac{2}{5} \cos t + \frac{1}{5} \sin t & -\frac{2}{5} \sin t - \frac{1}{5} \cos t \\ \cos t & \sin t \end{pmatrix}$$

3. Find a particular real solution to the nonhomogeneous equation by

$$x^{\star}(t) = X(t) \int X^{-1}(t)\, g(t)\, \mathrm{dt},$$

where g is the source term

$$g(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$

4. The general solution is

$$x(t) = x^{\star}(t) + X(t)\, C,$$

where C is a constant vector.