MATH 2352 Solution Sheet 08

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Created on April 20, 2015

[Problems] 6.4: 10, 12; 6.5: 3, 7; 6.6: 1(c), 7, 10, 17, 20;

6.4 - 10. For

$$
\begin{aligned}
y'' + y' + \frac{5}{4}y &= g(t); \\
y(0) &= 1, \\
y'(0) &= 0; \\
g(t) &= \begin{cases} \sin t, & 0 \leqslant t < \pi \\ 0, & t \geqslant \pi \end{cases}.
\end{aligned}
$$

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

Solution.

(a) By applying Laplace transform on the I.V.P,

$$
\begin{aligned}
s^2F(s) - s + sF(s) - 1 + \frac{5}{4}F(s) &= G(s) \\
F(s) &= \frac{G(s) + s + 1}{s^2 - s + \frac{5}{4}} \\
&= \frac{G(s) + s + 1}{\left(s - \frac{1}{2}\right)^2 + 1},
\end{aligned}
$$

where

$$
\begin{aligned}
G(s) &= \mathcal{L}\{g(t)\}(s) \\
&= \mathcal{L}\{[1 - H(t-\pi)]\sin t\}(s) \\
&= \mathcal{L}\{\sin t\} - \mathcal{L}\{H(t-\pi)\sin t\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
H(t-\pi)\sin t &= -H(t-\pi)\sin(t-\pi) \\
&= -\delta(t-\pi) \star \sin(t),
\end{aligned}
$$

we have

$$
\begin{aligned}
G(s) &= \mathcal{L}\{\sin t\} - \mathcal{L}\{H(t-\pi)\sin t\} \\
&= \mathcal{L}\{\sin t\} + \mathcal{L}\{\delta(t-\pi)\}\,\mathcal{L}\{\sin t\} \\
&= \frac{1}{s^2+1} + \frac{1}{s^2+1}\,e^{-\pi s}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F(s) &= \frac{G(s)+s+1}{s^2-s+\frac{5}{4}} \\
&= \frac{\frac{1}{s^2+1}+\frac{1}{s^2+1}\,e^{-\pi s}+s+1}{\left(s-\frac{1}{2}\right)^2+1} \\
&= \frac{1+e^{-\pi s}+(s+1)(s^2+1)}{(s^2+1)\left[\left(s-\frac{1}{2}\right)^2+1\right]} \\
&= \frac{4}{17}e^{-\pi s}\left[-\frac{4s-3}{\left(s-\frac{1}{2}\right)^2+1}+\frac{4s+1}{s^2+1}\right]+\frac{1}{17}\left[\frac{s+29}{\left(s-\frac{1}{2}\right)^2+1}+4\,\frac{4s+1}{s^2+1}\right].
\end{aligned}
$$

Then using inverse Laplace transform,

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{F(s)\} \\
&= \frac{4}{17}\,\mathcal{L}^{-1}\{e^{-\pi s}\}\star\mathcal{L}^{-1}\left\{-\frac{4s-3}{\left(s-\frac{1}{2}\right)^2+1}+\frac{4s+1}{s^2+1}\right\}+\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{s+29}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2+1}\right\} \\
&= \frac{4}{17}\,\delta(t-\pi)\star\left[-\mathcal{L}^{-1}\left\{\frac{4\left(s-\frac{1}{2}\right)-1}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2+1}\right\}\right]+\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{s-\frac{1}{2}+\frac{59}{2}}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2+1}\right\} \\
&= \frac{4}{17}\,\delta(t-\pi)\star\left[e^{t/2}\,(\sin t-4\cos t)+\sin t+4\cos t\right]+\frac{1}{17}\left[\frac{1}{2}\,e^{t/2}\,(59\sin t+2\cos t)\right]+\frac{4}{17}\,[\sin t+4\cos t] \\
&= \frac{4}{17}\,H(t-\pi)\left[e^{(t-\pi)/2}(4\cos t+\sin t)-\sin t-4\cos t\right]+\frac{1}{17}\left[e^{t/2}\left(\frac{59}{2}\sin t+\cos t\right)+4\sin t+4\cos t\right].
\end{aligned}
$$

(b) Since

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{G(s)}{\left(s-\frac{1}{2}\right)^2+1}+\frac{s+1}{\left(s-\frac{1}{2}\right)^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{G(s)}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s+1}{\left(s-\frac{1}{2}\right)^2+1}\right\} \\
&= \mathcal{L}^{-1}\{G(s)\}\star\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s+1}{\left(s-\frac{1}{2}\right)^2+1}\right\} \\
&= g(t)\star\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s+1}{\left(s-\frac{1}{2}\right)^2+1}\right\} \\
&= g(t)\star e^{t/2}\sin t+\phi(t),
\end{aligned}
$$

where

$$\phi(t) \;=\; e^{t/2}\left(\frac{3}{2}\sin t + \cos t\right)$$

is the solution to the initial value problem with homogeneous source term $g(t) = 0$.

Therefore, $g(s)$ has an influence on $y(t)$ with a weight $e^{(t-s)/2}\sin(t-s)$, $(s \leqslant t)$.

```
GNUplot] g(t) = t<pi ? sin(t) : 0;
         y(t) = t<pi ? (1.0/17)*(exp(t/2)*((59/2.0)*sin(t)+cos(t)+4*sin(t)+4*cos(t))) : (4.0/
         17)*(exp((t-pi)/2)*(4*cos(t)+sin(t))-sin(t)-4*cos(t))+(1.0/17)*(exp(t/2)*((59/
         2.0)*sin(t)+cos(t)+4*sin(t)+4*cos(t)));
         set yrange [-2:2];
         plot [0:10] g(x), y(x)/100
```

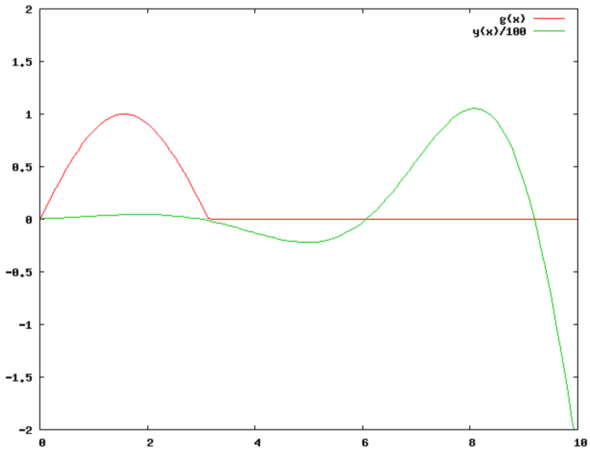

Figure 1.

6.4 - 12. For

$$\begin{aligned} y^{(4)} - y &= u_1(t) - u_3(t); \\ y(0) &= 0, \\ y'(0) &= 0, \\ y''(0) &= 0, \\ y'''(0) &= 0; \\ u_c(t) &= \begin{cases} 0, & t < c, \\ 1, & t \geqslant c. \end{cases} \end{aligned}$$

(a) Find the solution of the given initial value problem.

(b) Draw the graphs of the solution and of the forcing function; explain how they are related.

Solution.

(a) By applying Laplace transform to the IVP,

$$\begin{aligned}
s^4F(s)-F(s) &= \mathcal{L}\{H(t-1)\}-\mathcal{L}\{H(t-3)\}\\
&= \frac{1}{s}(e^{-s}-e^{-3s})\\
F(s) &= \frac{e^{-s}-e^{-3s}}{s(s-1)(s+1)(s^2+1)}\\
&= (e^{-s}-e^{-3s})\left[\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{F(s)\}(t)\\
&= \mathcal{L}^{-1}\left\{(e^{-s}-e^{-3s})\left[\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right]\right\}(t)\\
&= \mathcal{L}^{-1}\left\{e^{-s}\left[\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right]\right\}(t)-\mathcal{L}^{-1}\left\{e^{-3s}\left[\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right]\right\}(t)\\
&= \mathcal{L}^{-1}\{e^{-s}\}\star\mathcal{L}^{-1}\left\{\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right\}-\mathcal{L}^{-1}\{e^{-3s}\}\star\mathcal{L}^{-1}\left\{\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{L}^{-1}\{e^{-s}\} &= \delta(t-1),\\
\mathcal{L}^{-1}\{e^{-3s}\} &= \delta(t-3),\\
\mathcal{L}^{-1}\left\{\frac{s}{2(s^2+1)}+\frac{1}{4(s-1)}+\frac{1}{4(s+1)}-\frac{1}{s}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\\
&= \frac{1}{2}\cos t+\frac{1}{4}e^t+\frac{1}{4}e^{-t}-1,
\end{aligned}$$

we have

$$\begin{aligned}
y(t) &= \delta(t-1)\star\left[\frac{1}{2}\cos t+\frac{1}{4}e^t+\frac{1}{4}e^{-t}-1\right]-\delta(t-3)\star\left[\frac{1}{2}\cos t+\frac{1}{4}e^t+\frac{1}{4}e^{-t}-1\right]\\
&= H(t-1)\left[\frac{1}{2}\cos(t-1)+\frac{1}{4}e^{t-1}+\frac{1}{4}e^{-t+1}-1\right]-H(t-3)\left[\frac{1}{2}\cos(t-3)+\frac{1}{4}e^{t-3}+\frac{1}{4}e^{-t+3}-1\right].
\end{aligned}$$

(b) Since

$$F(s) = \frac{\mathcal{L}\{g(t)\}(s)}{s(s-1)(s+1)(s^2+1)},$$

we have

$$\begin{aligned}
y(t) &= g(t)\star\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)(s+1)(s^2+1)}\right\}\\
&= g(t)\star\left[\frac{1}{2}\cos t+\frac{1}{4}e^t+\frac{1}{4}e^{-t}-1\right].
\end{aligned}$$

That is, the solution is the convolution of the forcing function with a specific function determined by the equaiotion.

```
GNUplot] g(t) = t<1 ? 0 : t<3 ? 1 : 0;

         y(t) = t<1 ? 0 : t<3 ? (0.5*cos(t-1)+0.25*exp(t-1)+0.25*exp(-t+1)-1)
         : (0.5*cos(t-1)+0.25*exp(t-1)+0.25*exp(-t+1)-1) -
         (0.5*cos(t-3)+0.25*exp(t-3)+0.25*exp(-t+3)-1);

         plot [0:4] g(x), y(x)
```

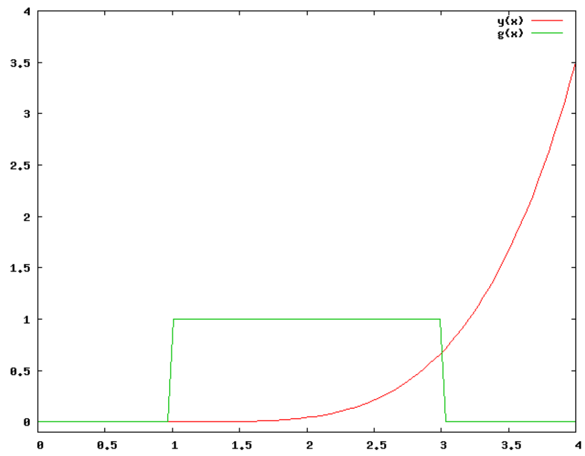

Figure 2.

6.5 - 3. For

$$\begin{aligned} y'' + 3y' + 2y &= \delta(t-\pi); \\ y(0) &= 1, \\ y'(0) &= 1. \end{aligned}$$

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

Solution.

(a) By Laplace transform,

$$\begin{aligned} (s^2+3s+2)F(s) - s - 1 - 3 &= e^{-\pi s} \\ F(s) &= \frac{e^{-\pi s}+s+4}{(s+1)(s+2)} \\ &= e^{-\pi s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right] + \left[\frac{3}{s+1} - \frac{2}{s+2}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \delta(t-\pi) \star [e^{-t} - e^{-2t}] + [3e^{-t} - 2e^{-2t}] \\ &= H(t-\pi)\,(e^{-t+\pi} - e^{-2t+2\pi}) + (3e^{-t} - 2e^{-2t}). \end{aligned}$$

(b)

```
GNUplot] y(t) = t<pi ? (3*exp(-t)-2*exp(-2*t)) :
         (exp(-t+pi)-exp(-2*t+2*pi))+(3*exp(-t)-2*exp(-2*t));
         plot [0:9] y(x);
```

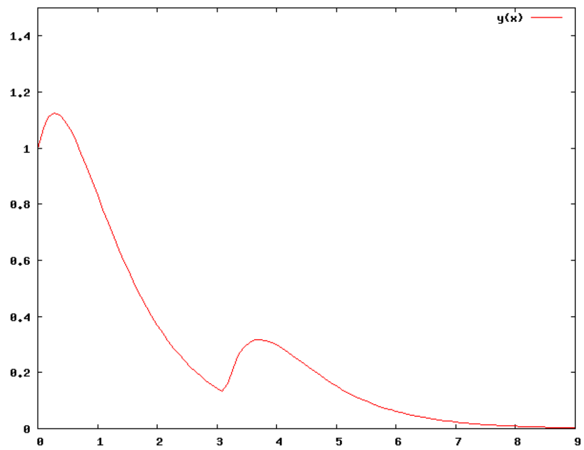

Figure 3.

6.5 - 7. For

$$
\begin{aligned}
y'' + y &= \delta(t-2\pi)\cos t; \\
y(0) &= 1, \\
y'(0) &= 1;
\end{aligned}
$$

(a) Find the solution of the given initial value problem.

(b) Draw a graph of the solution.

Solution.

(a) Since

$$
\begin{aligned}
\delta(t-2\pi)\cos t &= \begin{cases} 0, & t \neq 2\pi \\ \delta(0)\cos 2\pi, & t = 2\pi \end{cases} \\
&= \delta(t-2\pi)\cos 2\pi \\
&= \delta(t-2\pi).
\end{aligned}
$$

Using Laplace transform,

$$
\begin{aligned}
s^2F(s) - s - 1 + F(s) &= e^{-2\pi s} \\
F(s) &= \frac{e^{-2\pi s} + s + 1}{s^2+1} \\
&= e^{-2\pi s}\frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(t) &= \delta(t-2\pi) \star \sin t + \cos t + \sin t \\
&= H(t-2\pi)\sin t + \cos t + \sin t \\
&= \cos t + [1 + H(t-2\pi)]\sin t.
\end{aligned}
$$

(b)

```
GNUplot] y(t) = t<2*pi ? (cos(t)+sin(t)) : (cos(t)+2*sin(t));
         plot [0:15] y(x);
```

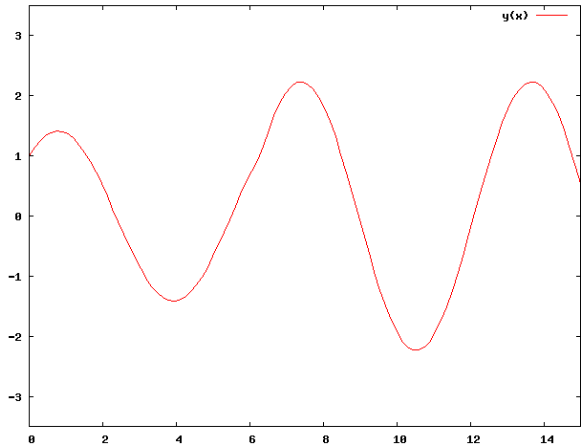

Figure 4.

6.6 - 1 (c). Prove the associative property of the convolution integral:

$$f \star (g \star h) \;=\; (f \star g) \star h.$$

Proof.

By definition,

$$\begin{aligned}
[f \star (g \star h)]\,(t) &= \int_{-\infty}^{\infty} f(t-\tau)(g \star h)(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} f(t-\tau)\left[\int_{-\infty}^{\infty} g(\tau - s)\,h(s)\,ds\right] d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t-\tau)g(\tau-s)\,h(s)\,ds\,d\tau.
\end{aligned}$$

Under the condition of Fubini's theorem, we have

$$\begin{aligned}
f \star (g \star h)(t) &= \int_{-\infty}^{\infty} h(s)\,ds \int_{-\infty}^{\infty} f(t-\tau)g(\tau-s)\,d\tau \\
&= \int_{-\infty}^{\infty} h(s)\,(f \star g)(t-s)\,ds \\
&= [(f \star g) \star h]\,(t).
\end{aligned}$$

□

6.6 - 7. Find the Laplace transform of

$$f(t) = \int_0^t \sin(t-\tau)\cos 3\tau d\tau.$$

Solution. By definition, using Fubini's theorem and integration by part,

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st}dt \int_0^t \sin(t-\tau)\cos 3\tau d\tau \\
&= -\frac{1}{s}\int_0^\infty de^{-st}\int_0^t \sin(t-\tau)\cos 3\tau d\tau \\
&= -\frac{1}{s}\left\{\left[e^{-st}\int_0^t \sin(t-\tau)\cos 3\tau d\tau\right]\Big|_0^\infty - \int_0^\infty e^{-st}\left[\sin(t-t)\cos 3t + \int_0^t \cos(t-\tau)\cos 3\tau d\tau\right]dt\right\} \\
&= \frac{1}{s}\int_0^\infty e^{-st}dt\int_0^t \cos(t-\tau)\cos 3\tau d\tau \\
&= -\frac{1}{s^2}\int_0^\infty de^{-st}\int_0^t \cos(t-\tau)\cos 3\tau d\tau \\
&= -\frac{1}{s^2}\left\{\left[e^{-st}\int_0^t \cos(t-\tau)\cos 3\tau d\tau\right]\Big|_0^\infty - \int_0^\infty e^{-st}\left[\cos(t-t)\cos 3t - \int_0^t \sin(t-\tau)\cos 3\tau d\tau\right]dt\right\} \\
&= \frac{1}{s^2}\int_0^\infty e^{-st}\left[\cos 3t - \int_0^t \sin(t-\tau)\cos 3\tau d\tau\right]dt \\
&= \frac{1}{s^2}\int_0^\infty e^{-st}\cos 3t dt - \frac{1}{s^2}\int_0^\infty e^{-st}dt\int_0^t \sin(t-\tau)\cos 3\tau d\tau \\
&= \frac{1}{s^2}\int_0^\infty e^{-st}\cos 3t dt - \frac{1}{s^2}\mathcal{L}\{f(t)\}(s).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \frac{\frac{1}{s^2}\int_0^\infty e^{-st}\cos 3t dt}{1+\frac{1}{s^2}} \\
&= \frac{\int_0^\infty e^{-st}\cos 3t dt}{s^2+1}.
\end{aligned}$$

And since

$$\begin{aligned}
\int_0^\infty e^{-st}\cos 3t dt &= \mathcal{L}\{\cos 3t\}(s) \\
&= \frac{s}{s^2+9},
\end{aligned}$$

we have

$$\mathcal{L}\{f(t)\}(s) = \frac{s}{(s^2+9)(s^2+1)}.$$

Solution. Or the more direct way (while not the simpler way), compute

$$\int_0^t \sin(t-\tau)\cos 3\tau d\tau = \frac{1}{2}\sin^2 t\cos t.$$

Then evaluate the integral

$$\begin{aligned}
\int_0^\infty e^{-st}\frac{1}{2}\sin^2 t\cos t dt &= \frac{1}{2}\int_0^\infty e^{-st}(1-\cos^2 t)\cos t dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}\cos t dt - \frac{1}{2}\int_0^\infty e^{-st}\cos^3 t dt.
\end{aligned}$$

And use integration by part three times to express the second term using $\mathcal{L}\{\cos t\}(s)$.

6.6 - 10. Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s+1)^3\,(s^2+4)}$$

by using the convolution theorem.

Solution. By convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)\star\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)\star\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)\star\frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t)\\
&= e^{-t}\star e^{-t}\star e^{-t}\star\frac{1}{2}\sin(2t)\\
&= \left[\int_0^t e^{-(t-\tau)}e^{-\tau}d\tau\right]\star e^{-t}\star\frac{1}{2}\sin(2t)\\
&= te^{-t}\star e^{-t}\star\frac{1}{2}\sin(2t)\\
&= \left[\int_0^t e^{-(t-\tau)}\tau e^{-\tau}d\tau\right]\star\frac{1}{2}\sin(2t)\\
&= \frac{1}{2}t^2e^{-t}\star\frac{1}{2}\sin(2t)\\
&= \frac{1}{4}\int_0^t \tau^2 e^{-\tau}\sin(2(t-\tau))d\tau.
\end{aligned}$$

Using integration by part mutiple times, one can obtain

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{10}t^2e^{-t}+\frac{4}{5}t-\frac{11}{250}e^t\sin(2t)+\frac{2}{250}e^t\cos(2t)-\frac{2}{250}.$$

6.6 - 17. Express the solution of the initial value problem

$$\begin{aligned}
y''+4y'+4y &= g(t);\\
y(0) &= 1,\\
y'(0) &= -2;
\end{aligned}$$

in terms of a convolution integral.

Solution. Using Laplace transform,

$$\begin{aligned}
s^2F(s)-s-2+4sF(s)-4+4F(s) &= \mathcal{L}\{g(t)\}(s)\\
F(s) &= \frac{\mathcal{L}\{g(t)\}(s)+s+6}{s^2+4s+4}\\
&= \frac{\mathcal{L}\{g(t)\}(s)}{(s+2)^2}+\frac{s+2+4}{(s+2)^2}\\
&= \frac{\mathcal{L}\{g(t)\}(s)}{(s+2)^2}+\frac{4}{(s+2)^2}+\frac{1}{s+2}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
y(t) &= g(t)\star\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}+4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\\
&= g(t)\star(te^{-2t})+(4t+1)e^{-2t}.
\end{aligned}
$$

6.6 - 20. Express the solution of the initial value problem

$$
\begin{aligned}
y^{(4)}+5y''+4y &= g(t);\\
y(0) &= 1,\\
y'(0) &= 0,\\
y''(0) &= 0,\\
y'''(0) &= 0;
\end{aligned}
$$

in terms of a convolution integral.

Solution. Using Laplace transform,

$$
\begin{aligned}
s^4F(s)-s^3+5s^2F(s)-5s+4F(s) &= \mathcal{L}\{g(t)\}(s)\\
F(s) &= \frac{\mathcal{L}\{g(t)\}(s)}{s^4+5s^2+4}+\frac{s^3+5s}{s^4+5s^2+4}\\
&= \frac{1}{3}\mathcal{L}\{g(t)\}(s)\left[\frac{1}{s^2+1}-\frac{1}{2}\frac{2}{s^2+4}\right]+\left[\frac{4}{3}\frac{s}{s^2+1}-\frac{1}{3}\frac{s}{s^2+4}\right].
\end{aligned}
$$

Therefore,

$$
y(t) = g(t)\star\left[\frac{1}{3}\sin t-\frac{1}{6}\sin(2t)\right]+\left[\frac{4}{3}\cos t-\frac{1}{3}\cos(2t)\right].
$$