MATH 2352 Solution Sheet 07

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[Problems] 6.1: 5(c), 14; 6.2: 2, 8, 14, 21; 6.3: 9, 11, 13, 20, 23, 25;

6.1 - 5. (c) Find the Laplace transform of each of the following functions

$$f(t) = t^n,$$

where n is a positive integer.

Solution. By definition, for $n \geqslant 1$, $s \neq 0$, and assume $s \in \mathbb{C}$ s.t. $\lim_{t\to\infty}(e^{-st}t^n) = 0$ holds for $\forall n$,

$$\begin{aligned}
\mathcal{L}[f_n(t)](s) &= \int_0^\infty e^{-st} f_n(t)\,dt \\
&= \int_0^\infty e^{-st} t^n\,dt \\
&= -s^{-1}\int_0^\infty t^n\,de^{-st} \\
&= -s^{-1}\Big[(t^n e^{-st})\Big|_0^\infty - n\int_0^\infty e^{-st} t^{n-1} dt\Big] \\
&= n\,s^{-1}\int_0^\infty e^{-st} f_{n-1}(t) dt \\
&= n\,s^{-1}\{\mathcal{L}[f_{n-1}(t)](s)\}.
\end{aligned}$$

For $n = 0$, $s \neq 0$,

$$\begin{aligned}
\mathcal{L}[f_0(t)](s) &= \int_0^\infty e^{-st} f_0(t)\,dt \\
&= \int_0^\infty e^{-st}\,dt \\
&= s^{-1}.
\end{aligned}$$

Therefore, for positive integer n, and $s \neq 0$,

$$\begin{aligned}
\mathcal{L}[f_n(t)](s) &= n!\,s^{-n}\,s^{-1} \\
&= n!\,s^{-n-1}.
\end{aligned}$$

For $s = 0$,

$$\begin{aligned}
\mathcal{L}[f_n(t)](0) &= \int_0^\infty f_n(t)\,dt \\
&= \int_0^\infty t^n\,dt \\
&= \infty.
\end{aligned}$$

And for s where $\lim_{t\to\infty} (e^{-st}t^n) \neq 0$, $\mathcal{L}[f_n(t)](s) = \infty$.

6.1 - 14. Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin(bt) = (e^{ibt} + e^{-ibt})/2i$. Find the Laplace transform of the given function

$$f(t) = e^{at}\cos bt,$$

where a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

Solution. By definition, for s s.t. $\cos(bt)\, e^{(a-s)t} \to 0, t\to\infty$ and $\sin(bt)\, e^{(a-s)t} \to 0, t\to\infty$,

$$\begin{aligned}
\mathcal{L}[f(t)](s) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\infty e^{-st} e^{at} \cos(bt)\, dt \\
&= \int_0^\infty e^{(a-s)t} \cos(bt)\, dt \\
&= \frac{1}{a-s} \int_0^\infty \cos(bt)\, d e^{(a-s)t} \\
&= \frac{1}{a-s} \left[\cos(bt)\, e^{(a-s)t}\right]\Big|_0^\infty + \frac{b}{a-s} \int_0^\infty \sin(bt)\, e^{(a-s)t}\, dt \\
&= -\frac{1}{a-s} + \frac{b}{(a-s)^2} \int_0^\infty \sin(bt)\, d e^{(a-s)t} \\
&= -\frac{1}{a-s} + \frac{b}{(a-s)^2} \left[\sin(bt)\, e^{(a-s)t}\right]\Big|_0^\infty - \frac{b^2}{(a-s)^2} \int_0^\infty \cos(bt)\, e^{(a-s)t}\, dt \\
&= -\frac{1}{a-s} - \frac{b^2}{(a-s)^2} \int_0^\infty \cos(bt)\, e^{(a-s)t}\, dt \\
&= -\frac{1}{a-s} - \frac{b^2}{(a-s)^2} \mathcal{L}[f(t)](s).
\end{aligned}$$

Then from the equation above,

$$\begin{aligned}
\mathcal{L}[f(t)](s) &= \frac{-\frac{1}{a-s}}{1 + \frac{b^2}{(a-s)^2}} \\
&= \frac{s-a}{(a-s)^2 + b^2}.
\end{aligned}$$

For s that does not satisfy the assumption above, $\mathcal{L}[f(t)](s) = \infty$.

6.2 - 2. Find the inverse Laplace transform of the given function

$$F(s) = \frac{5}{(s-1)^3}.$$

Solution. Via Post's inversion formula, since

$$F^{(k)}(s) = (-1)^k \frac{(k+2)!}{2} \frac{5}{(s-1)^{3+k}},$$

we have

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right) \\
&= \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} (-1)^k \frac{(k+2)!}{2} \frac{5}{(k/t-1)^{3+k}} \\
&= \lim_{k\to\infty} \frac{(k+2)(k+1)}{2} \frac{5\,(k/t)^{k+1}}{(k/t-1)^{3+k}} \\
&= \lim_{k\to\infty} \frac{(k+2)(k+1)}{2} \frac{5\,k^{k+1}\,t^2}{(k-t)^{3+k}} \\
&= \frac{5\,t^2}{2} \lim_{k\to\infty} \frac{k^{k+1}\,(k+2)(k+1)}{(k-t)^{3+k}} \\
&= \frac{5\,t^2}{2} \lim_{k\to\infty} \frac{(1+2/k)(1+1/k)}{(1-t/k)^{3+k}} \\
&= \frac{5\,t^2}{2} \lim_{k\to\infty} \frac{(1+2/k)(1+1/k)}{(1-t/k)^k\,(1-t/k)^3} \\
&= \frac{5\,e^t\,t^2}{2}.
\end{aligned}$$

6.2 - 8. Find the inverse Laplace transform of the given function

$$F(s) = \frac{8s^2-6s+12}{s\,(s^2+4)}.$$

Solution. Since

$$\begin{aligned}
F(s) &= \frac{8s^2-6s+12}{s\,(s^2+4)} \\
&= \frac{3\,(s^2+4)+5\,s^2-6s}{s\,(s^2+4)} \\
&= \frac{3}{s}+\frac{5\,s}{s^2+4}+\frac{-6}{s^2+4}.
\end{aligned}$$

By linearity,

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \mathcal{L}^{-1}\left\{\frac{3}{s}+\frac{5\,s}{s^2+4}+\frac{-6}{s^2+4}\right\}(t) \\
&= 3\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t)+5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t)-3\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t).
\end{aligned}$$

Then it is easy to calculate (or look up the table to get)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) &= 1, \\
\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}(t) &= \cos(2t), \\
\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}(t) &= \sin(2t).
\end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\{F(s)\}(t) = 3+5\cos(2t)-3\sin(2t).$$

6.2 - 14. Use Laplace transform to solve the given initial value problem

$$\begin{aligned} y'' - 4y' + 4y &= 0, \\ y(0) &= 1, \\ y'(0) &= 3. \end{aligned}$$

Solution. Apply Laplace transform on the ODE, noting

$$\begin{aligned} \mathcal{L}\{f'\} &= s\,\mathcal{L}\{f\} - f(0), \\ \mathcal{L}\{f''\} &= s^2\,\mathcal{L}\{f\} - s\,f(0) - f'(0). \end{aligned}$$

We have

$$\begin{aligned} s^2F(s) - s - 3 - 4(s\,F(s) - 1) + 4F(s) &= 0 \\ (s^2 - 4s + 4)F(s) - s + 1 &= 0 \\ F(s) &= \frac{s-1}{(s-2)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{1}{(s-2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} \\ &= e^{2t} + t\,e^{2t}. \end{aligned}$$

6.2 - 21. Use Laplace transform to solve the given initial value problem

$$\begin{aligned} y'' - 2y' + 2y &= \cos t, \\ y(0) &= 1, \\ y'(0) &= 1. \end{aligned}$$

Solution. Using Laplace transform, we have

$$\begin{aligned} s^2\,\mathcal{L}\{f\} - s - 1 - 2\,(s\,\mathcal{L}\{f\} - 1) + 2\,\mathcal{L}\{f\} &= \mathcal{L}\{\cos t\} \\ (s^2 - 2s + 2)F(s) - s + 1 &= \frac{s}{s^2+1} \\ F(s) &= \frac{\frac{s}{s^2+1} + s - 1}{s^2 - 2s + 2} \\ &= \frac{s^3 - s^2 + 2s - 1}{(s^2+1)(s^2-2s+2)}. \end{aligned}$$

Then by partial fraction expansion,

$$F(s) = \frac{s-2}{5(s^2+1)} + \frac{4s-1}{(s-1)^2+1}.$$

Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s-2}{5(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{4s-1}{5[(s-1)^2+1]}\right\} \\ &= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \\ &= \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5}e^t\cos t + \frac{3}{5}e^t\sin t. \end{aligned}$$

6.3 - 9. For

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 2, \\ e^{-2(t-2)}, & t \geqslant 2. \end{cases}$$

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

Solution.

(a)

```
GNUplot] set samples 1000;
         set yrange [0:1.1];
         f(x) = x>=2 ? exp(-2*(x-2)) : x>=0 ? 1 : 0;
         plot [0:3] f(x);
```

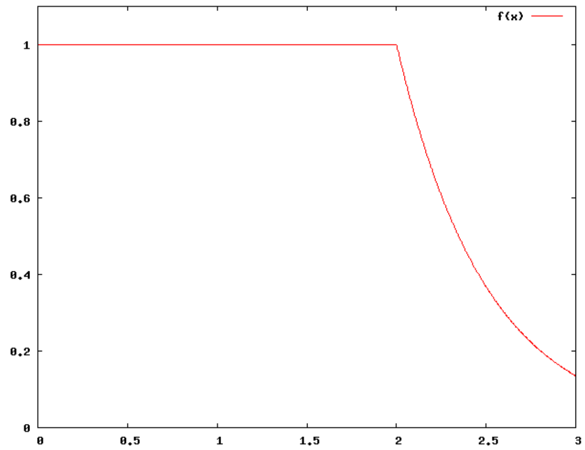

Figure 1.

(b) Define the unit step function

$$u_c(t) = \int_{-\infty}^{t} \delta(u)\,du.$$

Then

$$\begin{aligned} f(t) &= 1\,[1 - u_c(t-2)] + e^{-2(t-2)}\,u_c(t-2) \\ &= 1 + \left[e^{-2(t-2)} - 1\right]u_c(t-2)\,. \end{aligned}$$

6.3 - 11. For

$$f(t) = \begin{cases} t, & 0 \leqslant t < 1, \\ t-1, & 1 \leqslant t < 2, \\ t-2, & 2 \leqslant t < 3 \\ 0, & t \geqslant 3. \end{cases}$$

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

Solution.

(a)

```
GNUplot] set samples 1000;
         f(x) = x>=3 ? 0 : x>=2 ? x-2 : x>=1 ? x-1 : x>=0 ? x : 0;
         plot [0:4] f(x);
```

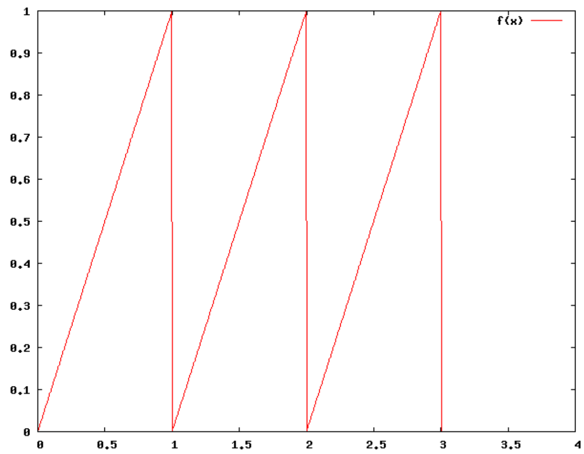

Figure 2.

(b) Define the unit step function

$$u_c(t) = \int_{-\infty}^{t} \delta(u)\, du.$$

Then

$$\begin{aligned} f(t) &= t + (-1)\, u_c(t-1) + (-1)\, u_c(t-2) + (-t+2)\, u_c(t-3) \\ &= t - u_c(t-1) - u_c(t-2) - (t-2)\, u_c(t-3). \end{aligned}$$

6.3 - 13. Find the Laplace transform of the given function.

$$f(t) = \begin{cases} 0, & t < 2, \\ (t-2)^3, & t \geqslant 2. \end{cases}$$

Solution. By definition,

$$\begin{aligned}
\mathcal{L}[f(t)](s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^2 e^{-st}\,0\,dt + \int_2^\infty e^{-st}(t-2)^3\,dt \\
&= \int_2^\infty e^{-st}(t-2)^3\,dt \\
&= e^{-\mathrm{st}} \frac{-s^3(t-2)^3 - 3s^2(t-2)^2 - 6s(t-2) - 6}{s^4}\Big|_2^\infty \\
&= e^{-2s}\frac{6}{s^4}.
\end{aligned}$$

6.3 - 20. Find the inverse Laplace transform of the given function.

$$F(s) = \frac{e^{-3s}}{s^2+s-2}.$$

Solution. Since

$$\begin{aligned}
F(s) &= \frac{e^{-3s}}{(s-1)(s+2)} \\
&= e^{-3s}\frac{1}{s-1}\frac{1}{s+2}.
\end{aligned}$$

Using convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s-1}\frac{1}{s+2}\right\} \\
&= \mathcal{L}^{-1}\{e^{-3s}\} \star \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \star \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= \delta(t-3) \star e^t \star e^{-2t}, \quad (t \geqslant 0).
\end{aligned}$$

Since

$$\begin{aligned}
e^t \star e^{-2t} &= \int_0^t e^{t-\tau} e^{-2\tau} d\tau \\
&= e^t \int_0^t e^{-3\tau} d\tau \\
&= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^t,
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\}(t) &= \int_0^t \delta(t-\tau-3)\,(e^t \star e^{-2t})(\tau) d\tau \\
&= \int_0^t \delta(t-\tau-3)\left[-\frac{1}{3}e^{-2\tau} + \frac{1}{3}e^{\tau}\right] d\tau \\
&= \left[-\frac{1}{3}e^{-2(t-3)} + \frac{1}{3}e^{(t-3)}\right] H(t-3) \\
&= \frac{1}{3}e^{-2(t-3)}\left[e^{3(t-3)} - 1\right] H(t-3),
\end{aligned}$$

where $H(t)$ is the Heaviside function.

6.3 - 23. Find the inverse Laplace transform of the given function.

$$F(s) = \frac{(s-2)\,e^{-2s}}{s^2-4s+3}.$$

Solution. Since

$$\begin{aligned} F(s) &= \frac{(s-2)\,e^{-2s}}{s^2-4s+3} \\ &= \frac{(s-2)\,e^{-2s}}{(s-2)^2-1} \\ &= \frac{(s-2)}{(s-2)^2-1}\,e^{-2s}. \end{aligned}$$

Using convolution theorem,

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2-1}\,e^{-2s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2-1}\right\}\star\mathcal{L}^{-1}\{e^{-2s}\} \\ &= \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2-1}\right\}\star\mathcal{L}^{-1}\{e^{-2s}\} \\ &= e^{2t}(\cosh t)\star\delta(t-2) \\ &= \frac{1}{2}e^{t}(e^{2t}+1)\star\delta(t-2) \\ &= \frac{1}{2}e^{t-2}(e^{2t-4}+1)\,H(t-2), \end{aligned}$$

where $H(t)$ is the Heaviside function.

6.3 - 25. Suppose that $F(s)=\mathcal{L}\{f(t)\}$ exists for $s>a\geqslant 0$.

(a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\Big(\frac{s}{c}\Big), \quad s>ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f\Big(\frac{t}{k}\Big).$$

(c) Show that if a and b are constants with $a>0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f\Big(\frac{t}{a}\Big).$$

Proof.

(a) By definition,

$$\begin{aligned}\mathcal{L}[f(ct)](s) &= \int_0^\infty e^{-st} f(ct)\,dt \\ &= \frac{1}{c}\int_0^\infty e^{-s/c\,(ct)} f(ct)\,d\,(ct) \\ &= \frac{1}{c}\int_0^\infty e^{-s/cu} f(u)\,du \\ &= \frac{1}{c}F\Big(\frac{s}{c}\Big).\end{aligned}$$

The condition for this to hold is $s/c > a$, or $s > ca$.

(b) Using the conclusion of (a) and uniqueness theorem,

$$\begin{aligned}\mathcal{L}\{f(ct)\} &= \frac{1}{c}F\Big(\frac{s}{c}\Big) \\ f(ct) &= \frac{1}{c}\mathcal{L}^{-1}\Big\{F\Big(\frac{s}{c}\Big)\Big\}.\end{aligned}$$

Let $c = \frac{1}{k}$, we have

$$\frac{1}{k}f\Big(\frac{t}{k}\Big) = \mathcal{L}^{-1}\{F(ks)\}.$$

(c) From Fourier-Mellin formula,

$$\begin{aligned}f(t) &:= \mathcal{L}^{-1}\{F(s)\}(t) \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{L}^{-1}\{F(as+b)\}(t) &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(as+b)\,ds \\ &= \frac{1}{a}\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{(as+b)\,(t/a)}e^{-bt/a}F(as+b)\,d\,(as+b) \\ &= \frac{e^{-bt/a}}{a}\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{(as+b)\,(t/a)}F(as+b)\,d\,(as+b) \\ &= \frac{e^{-bt/a}}{a}\mathcal{L}^{-1}\{F(s)\}\Big(\frac{t}{a}\Big).\end{aligned}$$

□