MATH 2352 Solution Sheet 06

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Created on April 17, 2015

[Problems] 5.5: 6, 11; 5.6: 10, 18;

5.5 - 6. For the equation

$$x^2\,y''+x\,y'+(x-2)\,y \;=\; 0.$$

(a) Show that the given differential equation has a regular singular point at $x=0$.

(b) Determine the indicial equation, the recurrence relation, and the roots of the indicial equation.

(c) Fine the series solution ($x>0$) corresponding to the larger root.

(d) If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

Solution.

(a) By definition, the equaion is equivalent to

$$y''+\frac{1}{x}y'+\frac{x-2}{x^2}y \;=\; 0.$$

Since

$$x\left(\frac{1}{x}\right) \;=\; 1,$$

$$x^2\left(\frac{x-2}{x^2}\right) \;=\; x-2$$

are both analytic at $x=0$, the given differential equation has a regular singular point at $x=0$.

(b) Assume the series solution is of the form ($a_0\neq 0$):

$$\begin{aligned} y(x) &= (x-0)^r\sum_{i=0}^{\infty} a_i\,(x-0)^i \\ &= x^r\sum_{i=0}^{\infty} a_i x^i \\ &= \sum_{i=0}^{\infty} a_i x^{i+r} \end{aligned}$$

then

$$\begin{aligned} (x-2)y(x) &= \sum_{i=0}^{\infty} a_i x^{i+r+1}-2\sum_{i=1}^{\infty} a_i x^{i+r}-2\,a_0x^r \\ &= -2\,a_0x^r+\sum_{i=1}^{\infty}(a_{i-1}-2\,a_i)x^{i+r} \\ x\,y'(x) &= \sum_{i=0}^{\infty} a_i(i+r)x^{i+r} \\ x^2\,y''(x) &= \sum_{i=0}^{\infty} a_i(i+r)(i+r-1)x^{i+r} \end{aligned}$$

Then for x^r terms, we have the indicial equation

$$\begin{aligned} -2\,a_0 + a_0(0+r) + a_0(0+r)(0+r-1) &= 0 \\ -2 + r + r(r-1) &= 0 \\ r^2 &= 2 \\ r &= \pm\sqrt{2}. \end{aligned}$$

For the following terms, we have the the recurrence relation

$$\begin{aligned} (a_{i-1} - 2\,a_i) + a_i(i+r) + a_i(i+r)(i+r-1) &= 0 \\ a_i &= a_{i-1}\frac{1}{2-(i+r)(1+i+r-1)} \\ &= a_{i-1}\frac{1}{2-(i+r)^2}. \end{aligned}$$

(c) For $r = \sqrt{2}$,

$$\begin{aligned} a_i &= a_{i-1}\frac{1}{2-\left(i+\sqrt{2}\right)^2} \\ &= a_{i-1}\frac{-1}{i^2 + 2\sqrt{2}i} \\ &= a_{i-1}\frac{-1}{i\left(i+2\sqrt{2}\right)}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_n &= a_{n-1}\frac{-1}{n\left(n+2\sqrt{2}\right)} \\ &= a_{n-2}\frac{-1}{n\left(n+2\sqrt{2}\right)}\frac{-1}{(n-1)\left(n-1+2\sqrt{2}\right)} \\ &= .... \\ &= a_0\frac{-1}{n\left(n+2\sqrt{2}\right)}\frac{-1}{(n-1)\left(n-1+2\sqrt{2}\right)}\cdots\frac{-1}{1\left(1+2\sqrt{2}\right)} \\ &= a_0\frac{(-1)^n\,\Gamma\left(2\sqrt{2}+1\right)}{\Gamma(n+1)\,\Gamma\left(n+2\sqrt{2}+1\right)}. \end{aligned}$$

Then

$$\begin{aligned} y(x) &= \sum_{i=0}^{\infty} a_i x^{i+r} \\ &= \sum_{n=0}^{\infty} a_0\frac{(-1)^n\,\Gamma\left(2\sqrt{2}+1\right)}{\Gamma(n+1)\,\Gamma\left(n+2\sqrt{2}+1\right)}x^{n+\sqrt{2}}. \end{aligned}$$

(d) For $r = -\sqrt{2}$,

$$a_n = a_0\frac{(-1)^n\,\Gamma\left(-2\sqrt{2}+1\right)}{\Gamma(n+1)\,\Gamma\left(n-2\sqrt{2}+1\right)}.$$

Therefore

$$\begin{aligned} y_-(x) &= \sum_{i=0}^{\infty} a_i x^{i+r} \\ &= \sum_{n=0}^{\infty} a_0\frac{(-1)^n\,\Gamma\left(-2\sqrt{2}+1\right)}{\Gamma(n+1)\,\Gamma\left(n-2\sqrt{2}+1\right)}x^{n-\sqrt{2}}. \end{aligned}$$

5.5 - 11. The Legendre equation of order α is

$$(1-x^2)\,y''-2\,x\,y'+\alpha\,(\alpha+1)\,y \;=\; 0.$$

The solution of the equation near the ordinary point $x=0$ was discussed in Problem 22 and 23 of Section 5.3. In Example 4 of Section 5.4, it was also shown that $x=\pm 1$ are regular singular points.

(a) Determine the indicial equation and its roots for the point $x=1$.

(b) Find a series solution in powers of $x-1$ for $x-1>0$.

Hint: Write $1+x=2+(x-1)$. Alternatively, make the change of variable $x-1=t$ and determine a series solution in power of t.

Solution.

(a) Assume the series solution is of the form ($a_0\neq 0$):

$$\begin{aligned} y(x) &= (x-1)^r\sum_{i=0}^{\infty} a_i\,(x-1)^i \\ &= \sum_{i=0}^{\infty} a_i(x-1)^{i+r}. \end{aligned}$$

then

$$\begin{aligned}
\alpha\,(\alpha+1)y(x) &= \alpha\,(\alpha+1)\sum_{i=0}^{\infty} a_i(x-1)^{i+r} \\
&= \sum_{i=0}^{\infty} \alpha\,(\alpha+1)a_i(x-1)^{i+r}. \\
-2\,x\,y'(x) &= -2\,(x-1+1)\sum_{i=0}^{\infty}(i+r)\,a_i(x-1)^{i+r-1} \\
&= -2\,(x-1)\sum_{i=0}^{\infty}(i+r)\,a_i(x-1)^{i+r-1}-2\sum_{i=0}^{\infty}(i+r)\,a_i(x-1)^{i+r-1} \\
&= -2\sum_{i=0}^{\infty}(i+r)\,a_i(x-1)^{i+r}-2\sum_{i=-1}^{\infty}(i+r+1)\,a_{i+1}(x-1)^{i+r} \\
&= -2\sum_{i=-1}^{\infty}[(i+r+1)\,a_{i+1}+(i+r)\,a_i](x-1)^{i+r}, \quad (a_{-1}=0). \\
(1-x^2)\,y''(x) &= (1+x)(1-x)\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r-2} \\
&= -(x-1)(x-1+1)\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r-2} \\
&= -(x-1)^2\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r-2}-(x-1)\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r-2} \\
&= -\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r}-\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r-1} \\
&= -\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x-1)^{i+r}-\sum_{i=-1}^{\infty} a_{i+1}(i+r+1)(i+r)(x-1)^{i+r} \\
&= -\sum_{i=-1}^{\infty}\left[a_i(i+r)(i+r-1)+a_{i+1}(i+r+1)(i+r)\right](x-1)^{i+r}.
\end{aligned}$$

Then for lowest order x^{r-1} terms, we have the indicial equation

$$-2\,r\,a_0 - r(r-1)\,a_0 \;=\; 0.$$

Then by assumption $a_0 \neq 0$, so

$$\begin{aligned} r(r-1)+2r &= 0 \\ r^2+r &= 0 \\ r_1=0, \qquad & r_2=-1. \end{aligned}$$

(b) For general terms ($i \geqslant 0$), we have the recurrence relation

$$\begin{aligned} \alpha\,(\alpha+1)a_i - 2[(i+r+1)\,a_{i+1} + (i+r)\,a_i] - [a_i(i+r)(i+r-1) + a_{i+1}(i+r+1)(i+r)] &= 0 \\ a_i[\alpha\,(\alpha+1) - 2(i+r) - (i+r)(i+r-1)] + a_{i+1}[-2(i+r+1) - (i+r+1)(i+r)] &= 0 \\ a_i[\alpha\,(\alpha+1) - (i+r)(i+r+1)] &= a_{i+1}(i+r+1)(i+r+2) \end{aligned}$$

Therefore,

$$\begin{aligned} a_{i+1} &= \frac{\alpha\,(\alpha+1) - (i+r)(i+r+1)}{(i+r+1)(i+r+2)}\,a_i \\ &= \frac{(\alpha+i+r)(\alpha-i-r) + (\alpha-i-r)}{(i+r+1)(i+r+2)}\,a_i \\ &= \frac{(\alpha+i+r+1)(\alpha-i-r)}{(i+r+1)(i+r+2)}\,a_i \\ &= \frac{(\alpha+r+1+i)\ldots(\alpha+r+1)(\alpha-r)\ldots(\alpha-r-i)}{(r+1+i)\ldots(r+1)(r+2+i)\ldots(r+2)}\,a_0 \\ &= \frac{\Gamma(\alpha+r+2+i)\,\Gamma(\alpha-r+1)\,\Gamma(r+1)\,\Gamma(r+2)}{\Gamma(\alpha+r+1)\,\Gamma(\alpha-r-i)\,\Gamma(r+2+i)\,\Gamma(r+3+i)}\,a_0 \end{aligned}$$

For the larger root $r=0$,

$$\begin{aligned} a_{i+1} &= \frac{\Gamma(\alpha+2+i)\,\Gamma(\alpha+1)\,\Gamma(1)\,\Gamma(2)}{\Gamma(\alpha+1)\,\Gamma(\alpha-i)\,\Gamma(2+i)\,\Gamma(3+i)}\,a_0 \\ &= \frac{\Gamma(\alpha+2+i)\,\Gamma(\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha-i)\,[(i+1)!]^2(i+2)}\,a_0 \end{aligned}$$

Thus

$$\begin{aligned} y &= \sum_{i=0}^{\infty} a_i(x-1)^{i+r} \\ &= a_0\sum_{i=0}^{\infty} \frac{\Gamma(\alpha+1+i)\,\Gamma(\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha+1-i)\,i!\,(i+1)!}(x-1)^{i+r}. \end{aligned}$$

Note that for $r=-1$,

$$a_{i+1} \;=\; \frac{\Gamma(\alpha+1+i)\,\Gamma(\alpha+2)\,\Gamma(0)\,\Gamma(1)}{\Gamma(\alpha+r+1)\,\Gamma(\alpha-r-i)\,\Gamma(r+2+i)\,\Gamma(r+3+i)}\,a_0$$

It is not meaningful because $\Gamma(0)=\infty$. Or the first recurrence relation yields $a_1=\infty\,a_0$, which in return means

$$\begin{aligned} a_0 &= \frac{a_1}{\infty} \\ &= 0, \end{aligned}$$

and this contradicts our assumption.

5.6 - 10. For the equation

$$(x-2)^2\,(x+2)\,y'' + 4x\,y' + 3(x-2)\,y \;=\; 0.$$

(a) Find all the regular points of the given differential equation.

(b) Determine the indicial equation and the exponents at the singularity for each regular singular point.

Solution.

(a) All singular points:

$$x \;=\; 2, -2.$$

For $x = -2$,

$$(x+2)\frac{4\,x}{(x-2)^2(x+2)} \;=\; \frac{4x}{(x-2)^2},$$

$$(x+2)^2\frac{3(x-2)}{(x-2)^2(x+2)} \;=\; \frac{3(x+2)}{x-2},$$

they are both analytic at the point $x = -2$. Therefore, $x = -2$ is a regular singular point.

For $x = 2$,

$$(x-2)\frac{4x}{(x-2)^2(x+2)} \;=\; \frac{4x}{(x-2)(x+2)}$$

is not analytic at this point. Therefore, $x = 2$ is an irregular singular point.

(b) For the only regular singular point $x = -2$, assume the solution around this point has the form ($a_0 \neq 0$):

$$\begin{aligned} y(x) &= (x+2)^r \sum_{i=0}^{\infty} a_i\,(x+2)^i \\ &= \sum_{i=0}^{\infty} a_i(x+2)^{i+r}. \end{aligned}$$

then

$$\begin{aligned} 3(x-2)\,y &= 3(x+2-4)\sum_{i=0}^{\infty} a_i(x+2)^{i+r} \\ &= 3\sum_{i=0}^{\infty} a_i(x+2)^{i+r+1} - 12\sum_{i=0}^{\infty} a_i(x+2)^{i+r} \\ 4x\,y' &= 4(x+2-2)\sum_{i=0}^{\infty} a_i(i+r)(x+2)^{i+r-1} \\ &= 4\sum_{i=0}^{\infty} a_i(i+r)(x+2)^{i+r} - 8\sum_{i=0}^{\infty} a_i(i+r)(x+2)^{i+r-1} \\ (x-2)^2\,(x+2)\,y'' &= (x+2-4)^2\,(x+2)\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x+2)^{i+r-2} \\ &= [(x+2)^2 - 8(x+2) + 16]\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x+2)^{i+r-1} \\ &= \sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x+2)^{i+r+1} - 8\sum_{i=0}^{\infty} a_i(i+r)(i+r-1)(x+2)^{i+r} + 16\sum_{i=0}^{\infty} a_i(i+r)(i+ \\ &\quad r-1)(x+2)^{i+r-1} \end{aligned}$$

For the lowest order $(x+2)^{r-1}$ term, we have the indicial equaion:

$$\begin{aligned}
-8a_0 r + 16a_0 r(r-1) &= 0 \\
2r(r-1) - r &= 0 \\
r(r-2) &= 0.
\end{aligned}$$

Then the roots are the exponents

$$\begin{aligned}
r_1 &= 0, \\
r_2 &= 2.
\end{aligned}$$

5.6 - 18.

(a) Show that

$$(\ln x)\, y'' + \frac{1}{2}\, y' + y = 0$$

has a regular singular point at $x = 1$.

(b) Determine the roots of the indicial equation at $x = 1$.

(c) Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n\, (x-1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$.

(d) What would you expect the radius of convergence of the series to be?

Solution.

(a) Since $\ln 1 = 0$, $x = 1$ is a singular point. And

$$\begin{aligned}
(x-1)\frac{1}{2\ln x} &= \frac{x-1}{2\ln\,[1 + (x-1)]} \\
&= \frac{x-1}{2[(x-1) + O(x-1)^2]}
\end{aligned}$$

is analytic at $x = 1$. Therefore, $x = 1$ is a regular singular point.

(b) Assume the solution around this point has the form ($a_0 \neq 0$):

$$\begin{aligned}
y(x) &= (x-1)^r \sum_{i=0}^{\infty} a_i\, (x-1)^i \\
&= \sum_{i=0}^{\infty} a_i (x-1)^{i+r}.
\end{aligned}$$

Then since

$$\begin{aligned}
\ln x &= \ln\,(1 + x - 1) \\
&= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (x-1)^i,
\end{aligned}$$

we have

$$\begin{aligned}
y &= \sum_{i=0}^{\infty} a_i (x-1)^{i+r} \\
\frac{1}{2}\,y' &= \frac{1}{2}\sum_{i=0}^{\infty} a_i (i+r)(x-1)^{i+r-1} \\
&= \frac{1}{2}\sum_{i=-1}^{\infty} a_{i+1}(i+1+r)(x-1)^{i+r} \\
(\ln x)\,y'' &= \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}(x-1)^i\right]\left[\sum_{i=0}^{\infty} a_i (i+r)(i+r-1)(x-1)^{i+r-2}\right] \\
&\quad \text{(assume absolute uniform convergence)} \\
&= \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{j+1}}{j} a_i (i+r)(i+r-1)(x-1)^{i+j+r-2} \\
&= \sum_{k=1}^{\infty}\left[\sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} a_{k-j}(k-j+r)(k-j+r-1)\right](x-1)^{r+k-2} \\
&= \sum_{i=-1}^{\infty}\left[\sum_{j=1}^{i+2} \frac{(-1)^{j+1}}{j} a_{i+2-j}(i-j+r+2)(i-j+r+1)\right](x-1)^{i+r}
\end{aligned}$$

For the lowest order $(x-1)^{r-1}$ terms, we have the indicial equation

$$\begin{aligned}
\frac{1}{2}\,a_0\,r + a_0\,r(r-1) &= 0 \\
r + 2r(r-1) &= 0 \\
r(2r-1) &= 0.
\end{aligned}$$

Then the roots are the exponents

$$\begin{aligned}
r_1 &= 0, \\
r_2 &= \frac{1}{2}.
\end{aligned}$$

(c) For the largest root $r = \frac{1}{2}$, we have the recurrence relation

$$a_i + \frac{1}{2} a_{i+1}\left(i+\frac{3}{2}\right) + \sum_{j=1}^{i+2} \frac{(-1)^{j+1}}{j} a_{i+2-j}\left(i-j+\frac{5}{2}\right)\left(i-j+\frac{3}{2}\right) = 0.$$

It determins a_{i+1} from $a_0, \ldots, a_i$. For $i=0$,

$$\begin{aligned}
a_0 + \frac{3}{4} a_1 + \sum_{j=1}^{2} \frac{(-1)^{j+1}}{j} a_{2-j}\left(-j+\frac{5}{2}\right)\left(-j+\frac{3}{2}\right) &= 0 \\
a_0 + \frac{3}{4} a_1 + \frac{(-1)^2}{1} a_1\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + \frac{(-1)^3}{2} a_0\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) &= 0 \\
a_0\left(1+\frac{1}{8}\right) + a_1\left(\frac{3}{4}+\frac{3}{4}\right) &= 0 \\
\frac{3}{2}\,a_1 &= -\frac{9}{8}\,a_0 \\
a_1 &= -\frac{3}{4}\,a_0.
\end{aligned}$$

For $i=1$,

$$a_1+\frac{5}{4}a_2+\sum_{j=1}^{3}\frac{(-1)^{j+1}}{j}a_{3-j}\left(-j+\frac{7}{2}\right)\left(-j+\frac{5}{2}\right) = 0$$

$$a_1+\frac{5}{4}a_2+\frac{(-1)^2}{1}a_2\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)+\frac{(-1)^3}{2}a_1\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)+\frac{(-1)^4}{3}a_0\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0$$

$$\frac{-1}{12}a_0+\left(1-\frac{3}{8}\right)a_1+\left(\frac{5}{4}+\frac{15}{4}\right)a_2 = 0$$

$$\begin{aligned} a_2 &= \frac{\frac{1}{12}a_0-\frac{5}{8}a_1}{5} \\ &= \frac{1}{5}\left(\frac{1}{12}-\left(-\frac{3}{4}\right)\frac{5}{8}\right)a_0 \\ &= \frac{53}{480}a_0. \end{aligned}$$

Therefore, the first three terms are $a_0(x-1)^r$, $-\frac{3}{4}a_0(x-1)^{1+r}$, $\frac{53}{480}a_0(x-1)^{2+r}$, where a_0 is an arbitary real constant.

(d) A reasonable expectation is 1 because the equaion has an irregular singular point at $x=0$.

In fact, using the relation

$$a_i+\frac{1}{2}a_{i+1}\left(i+\frac{3}{2}\right)+\sum_{j=1}^{i+2}\frac{(-1)^{j+1}}{j}a_{i+2-j}\left(i-j+\frac{5}{2}\right)\left(i-j+\frac{3}{2}\right) = 0$$

Let $k=i+2-j$ in the summation

$$a_i+\frac{1}{2}a_{i+1}\left(i+\frac{3}{2}\right)+\sum_{k=0}^{i+1}\frac{(-1)^{i+1-k}}{i+2-k}a_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right) = 0$$

$$(i+1)\left(i+\frac{3}{2}\right)a_{i+1}+\frac{1}{2}\left(\frac{\sqrt{7}}{2}+i\right)\left(\frac{\sqrt{7}}{2}-i\right)a_i+\sum_{k=0}^{i-1}\frac{(-1)^{i+1-k}}{i+2-k}a_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right) = 0$$

Therefore,

$$\begin{aligned} a_{i+1} &= \frac{\frac{1}{2}\left(i^2-\frac{7}{4}\right)a_i+\sum_{k=0}^{i-1}\frac{(-1)^{i-k}}{i+2-k}a_k\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{(i+1)\left(i+\frac{3}{2}\right)} \\ &= \frac{1}{2}\frac{\left(i^2-\frac{7}{4}\right)}{(i+1)\left(i+\frac{3}{2}\right)}a_i+\sum_{k=0}^{i-1}\frac{(-1)^{i-k}}{i+2-k}a_k\frac{\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{(i+1)\left(i+\frac{3}{2}\right)} \end{aligned}$$

When i is very large,

$$a_{i+1} = \frac{1}{2}a_i-\frac{1}{3}a_{i-1}+\frac{1}{4}a_{i-2}-\ldots+\frac{1}{N+2}a_{i-N}+\epsilon$$

where

$$N = N(i,\epsilon)$$

is an integer. Assume the ratio a_{i+1}/a_i converges to $0<c<\infty$ (c is positive because at least we have the leading term $a_{i+1}=\frac{1}{2}a_i+\ldots$), then let

$$\begin{aligned} a_{i-N} &= b \\ a_{i-N+1} &= cb \\ &\ldots \\ a_i &= c^N b \end{aligned}$$

Then approximately

$$c^{N+1} - \frac{1}{2}c^{N} + \frac{1}{3}c^{N-1} + ... + \frac{1}{N+1}c + \frac{1}{N+2} \;=\; 0$$

$$1 - \frac{1}{2}c^{-1} + \frac{1}{3}c^{-2} + ... + \frac{1}{N+1}c^{-N} + \frac{1}{N+2}c^{-N-1} \;=\; 0$$

The left hand side is the trancated Taylor expansion of $\ln c^{-1}$, so in when $i \to \infty$

$$\ln c^{-1} \;=\; 0$$

$$c \;=\; 1$$

Therefore, the radius of convergence is 1.