MATH 2352 Solution Sheet 02

BY XIAOYU WEI

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[Problems] 2.2: 3, 7, 16(a,c); 2.3: 2, 9; 2.4: 4, 6, 13; 2.6: 7, 9

2.2 - 3. Solve the given differential equation

$$y' + y^2 \cos x = 0.$$

Solution. Multiply both sides by y^{-2}, we have

$$\frac{y'}{y^2} + \cos x = 0.$$

It is noted that $\left(\frac{1}{y}\right)' = -\frac{y'}{y^2}$, so we have

$$\left(\frac{1}{y}\right)' = \cos x.$$

Then $y(x) = \frac{1}{\sin x + C}$ solves the original equation.

2.2 - 7. Solve the given differential equation

$$\frac{dy}{dx} = \frac{3x - e^{-x}}{2y + e^y}.$$

Solution. Rearrange the terms,

$$(2y + e^y)\, dy = (3x - e^{-x})\, dx.$$

Integrate both sides,

$$y^2 + e^y = \frac{3}{2}x^2 + e^{-x} + C,$$

which is the implicit representation of the solution.

2.2 - 16. For the equation

$$y' = x(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2}.$$

(a) Find the solution of the given initial value problem in explicit form.

(c) Determine (at least approximately) the interval in which the solution is defined.

Solution. Multiply both sides by $4y^3$,

$$4\, y'\, y^3 = x(x^2 + 1).$$

Observe that $(y^4)' = 4\, y^3\, y'$, therefore,

$$\begin{aligned} y^4 &= \int x(x^2+1)\, dx \\ &= \frac{1}{4} x^4 + \frac{1}{2} x^2 + C. \end{aligned}$$

From initial condition, $1/4 = C$

Then $y(x) = -\sqrt[4]{\frac{1}{4} x^4 + \frac{1}{2} x^2 + \frac{1}{4}} = -\sqrt{\frac{x^2+1}{2}}$. (Note the sign).

The solution is defined for $x \in \mathbb{R}$.

2.3 - 2. A tank initially contains $180\,(L)$ of pure water. A mixture containing a concentration of $\gamma\,(g/L)$ of salt enters the tank at a rate of $3\,(L/\mathrm{min})$, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution. Denote the amount of salt in the tank at time t as $y(t)$, we have

$$\begin{aligned} y' &= 3\gamma - \frac{y}{180} \times 3 \\ &= 3\gamma - \frac{1}{60}\, y, \end{aligned}$$

subject to initial condition $y(0) = 0$.

Solving the equation yields

$$y(t) = -180\,\gamma\, e^{-\frac{t}{60}} + 180\,\gamma.$$

As $t \to \infty$, $y(t) \to 180\,\gamma$, which means that the concentration of salt in the tank equals to the concentration of in flow.

2.3 - 9. A certain college graduate borrows \$9000 to buy a car. The lender charges interest at an annual rate of 8%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

Solution.

Denote $y(t)$ as the amount of money the guy owes the lender at time t. $t = 0$ when the borrowing happens. Then we have

$$y' = 0.08\, y - k,$$

subject to initial condition $y(0) = 9000$. The solution is

$$y(t) = (9000 - 12.5k)\, e^{0.08\,t} + 12.5k.$$

"Paying off in 3 years" means $y(3) = 0$, which yields $k = \frac{9000}{12.5\,(1 - e^{-0.24})} \approx 3374.4$ (USD).

The total interest is equal to $3k - 9000 \approx 1123.2$ (USD).

Remark 1. (Quoted from Wikipedia) Continuous compounding can be thought of as making the compounding period infinitesimally small, achieved by taking the limit as n goes to infinity. The amount after t periods of continuous compounding can be expressed in terms of the initial amount A_0 as

$$A(t) = A_0\, e^{r\,t}.$$

It has been shown that the mathematics of continuous compounding is not limited to the valuation of continuously compounded financial instruments and flow annuities, but rather that the exponential equation is a versatile model that may be used for valuation of all financial contracts normally encountered.

2.4 - 4. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

$$(16-t^2)\,y'+2t\,y=3t^2,\qquad y(-5)=1.$$

Solution. From the eqaution, we have

$$\begin{aligned} y' &= \frac{3t^2-2t\,y}{(4+t)(4-t)} := f(t,y). \\ \frac{\partial f}{\partial y} &= \frac{-2t}{(4+t)(4-t)}. \end{aligned}$$

Then the solution of the given initial value problem is certain to exist in $(-\infty,-4)$.

2.4 - 6. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

$$(\ln t)\,y'+y=\cot t,\qquad y(3)=3.$$

Solution. From the eqaution, we have

$$\begin{aligned} y' &= \frac{\cot t-y}{\ln t} := f(t,y). \\ \frac{\partial f}{\partial y} &= -\frac{1}{\ln t}. \end{aligned}$$

Since $3\in(1,\pi)$, the solution of the given initial value problem is certain to exist in $(1,\pi)$.

2.4 - 13. Solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y'=-2t\,/\,y,\qquad y(0)=y_0.$$

Solution. The general solution is $y(t)=\pm\sqrt{2\,(C-x^2)}$. The sign and constant C depends on y_0, and the (real) solution exists in the interval $\left[-\sqrt{C},\sqrt{C}\right]$. Since

$$|y(0)| \;=\; \sqrt{2C}=|y_0|,$$

we have $\sqrt{C}=\frac{|y_0|}{\sqrt{2}}$. Therefore, the larger $|y_0|$ is, the larger the interval in which the solution exists will be.

2.6 - 7. Determine whether the equation is exact. If it is exact, find the solution.

$$(e^x\sin y-3y\sin x)+(e^x\cos y+3\cos x)\,y' \;=\; 0.$$

Solution. Denote $I(x,y)=e^x\sin y-3y\sin x,\ J(x,y)=e^x\cos y+3\cos x$. The equation is equivalent to

$$I(x,y)\,dx+J(x,y)\,dy \;=\; 0.$$

Since I and J are continuously differentiable on $\mathbb{R}^2$, and

$$\frac{\partial I(x,y)}{\partial y} = e^x \cos y - 3\sin x = \frac{\partial J(x,y)}{\partial x},$$

the equation is exact.

A potential function $F(x,y)$ for the differential equation is

$$F(x,y) = e^x \sin y + 3y\cos x.$$

Then the solution is of the form $F(x,y) = C$, that is,

$$e^x \sin y + 3y\cos x = C.$$

2.6 - 9. Determine whether the equation is exact. If it is exact, find the solution.

$$(y\,e^{x\,y}\cos 2x - 2\,e^{x\,y}\sin 2x + 2x) + (x\,e^{x\,y}\cos 2x - 3)\,y' = 0.$$

Solution. Denote $I(x,y) = y\,e^{x\,y}\cos 2x - 2\,e^{x\,y}\sin 2x + 2x$, $J(x,y) = x\,e^{x\,y}\cos 2x - 3$. The equation is equivalent to

$$I(x,y)\,dx + J(x,y)\,dy = 0.$$

Since I and J are continuously differentiable on $\mathbb{R}^2$, and

$$\frac{\partial I(x,y)}{\partial y} = e^{x\,y}\cos 2x + x\,y\,e^{x\,y}\cos 2x - 2x\,e^{x\,y}\sin 2x = \frac{\partial J(x,y)}{\partial x},$$

the equation is exact.

A potential function $F(x,y)$ for the differential equation is

$$F(x,y) = e^{x\,y}\cos 2x + x^2 - 3y.$$

Then the solution is of the form $F(x,y) = C$, that is,

$$e^{x\,y}\cos 2x + x^2 - 3y = C.$$