MATH 2352 Solution Sheet 01

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1.2: 2c, 3, 7

1.3: 1-6, 18, 20, 21-24

2.1: 6, 8, 16

1.2 - 2 - c. Solve the following initial value problems and plot the solutions for several values of y0.

$$dy/dt = 2y - 6, \quad y(0) = y_0.$$

Solution. $y(t) = y_0 e^{2t} + 3$.

1.2 - 3. Consider the differential equation

$$dy/dt = -a\,y + b,$$

where both a and b are positive numbers.

(a) Find the general solution of the differential equation.

(b) Sketch the solution for several different initial conditions.

(c) Describe how the solutions change under each of the following conditions:

i. a increases.

ii. b increases.

iii. Both a and b increase, but the ratio b/a remains the same.

Solution. (a) $y(t) = C e^{-ax} - \frac{b}{a}$.

1.2 - 7. The field mouse population in Example 1 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 800$.

(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

(c) Find the initial population p_0 if the population is to become extinct in 1 year.

Solution. (a) $p(t) = [p(0) - 900]\,e^{\frac{t}{2}} + 900 = -100\,e^{\frac{t}{2}} + 900$, so $t|_{p(t)=0} = 4\ln 3$.

(b) $p(t) = (p_0 - 900)\,e^{\frac{t}{2}} + 900$, so $t|_{p(t)=0} = -2\ln\left(1 - \frac{p_0}{900}\right)$.

(c) becoming extinct in 1 year -900means $p(1) \leqslant 0$, that is,

$$p(1) = (p_0 - 900)\,e^{\frac{1}{2}} + 900 \leqslant 0, \text{ so } p_0 \leqslant -\frac{900}{\sqrt{e}} + 900 = 900\,\frac{e - \sqrt{e}}{e}.$$

1.3 - (1~6). In each of Problems 1 through 6, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2 \frac{d^2y}{dt^2} + t \frac{dy}{dt} + 3y = \sin t$
(order 2, linear)

2. $(1+y^2) \frac{d^2y}{dt^2} + t \frac{dy}{dt} + y = e^t$
(order 2, nonlinear)

3. $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$
(order 4, linear)

4. $\frac{dy}{dt} + t y^3 = 0$
(order 1, nonlinear)

5. $\frac{d^2y}{dt^2} + \cos(t+y) = \sin t$
(order 2, nonlinear)

6. $\frac{d^3y}{dt^3} + t \frac{dy}{dt} + (\sin^2 t) y = t^3$
(order 3, linear)

1.3 - 18. Determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

$$y''' - 4y'' + 3y' = 0.$$

Solution. plugging in $y = e^{rt}$ yields

$$r(r-3)(r-1)y = 0,$$

so $r \in \{0, 1, 3\}$.

1.3 - 20. Determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

$$t^2 y'' - 4t y' + 6y = 0.$$

Solution. plugging in $y = t^r$ yields

$$(r-2)(r-3)y = 0,$$

so $r \in \{2, 3\}$.

1.3 - (21~24). In each of Problems 21 through 24, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

21. $u_{xx} + u_{yy} + u_{zz} = 0$
(order 2, linear)

22. $u_{xx} + u_{yy} + u u_x + u_y + u = 0$
(order 2, nonlinear)

23. $u_{xxxx} + 4u_{xxyy} + u_{yyyy} = 0$
(order 4, linear)

24. $u_t + u u_x = 4 + u_{xx}$
(order 2, nonlinear)

2.1 - (6,8). For equation

$$t y' + 2y = 2\sin t, \quad t > 0. \tag{1}$$

And

$$(1+t^2) y' + 4t y = (1+t^2)^{-2} \tag{2}$$

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large t.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

Solution. for (c),

(1) for the first equation, multiply both sides by t,

$$(t^2\, y)' = 2\, t \sin t.$$

take integration with integration by part,

$$t^2\, y = \int 2\, t \sin t\, \mathrm{dt} = -2t \cos t + 2 \sin t + C.$$

therefore,

$$y(t) = \frac{-2t \cos t + 2 \sin t + C}{t^2}.$$

as $t \to \infty$, $y(t) \to 0$.

(2) for the second equation, multiply both sides by $(1 + t^2)$,

$$(1+t^2)^2\, y' + 4\,(1+t^2)\, t\, y = (1+t^2)^{-1},$$

or equivalently,

$$[(1+t^2)^2\, y]' = (1+t^2)^{-1}.$$

so

$$(1+t^2)^2\, y = \int (1+t^2)^{-1} \mathrm{dt} = \arctan(t) + C.$$

therefore,

$$y(t) = \frac{\arctan(t) + C}{(1+t^2)^2}.$$

as $t \to \infty$, $y(t) \to 0$.

2.1 - 16. Find the solution of the given initial value problem:

$$y' + \frac{3}{t}\, y = \frac{\cos t}{t^3}, \quad y(\pi) = 0, \quad t > 0.$$

Solution.

multiply both sides by t^3,

$$(t^3\, y)' = \cos t.$$

so

$$t^3\, y = \sin t + C.$$

therefore

$$y(t) = \frac{\sin t + C}{t^3}.$$

by $y(\pi) = 0$,

$$0 = y(\pi) = \frac{\sin \pi + C}{\pi^3},$$

we have $C = 0$, so

$$y(t) = \frac{\sin t}{t^3}.$$