MATH 2352 Problem Sheet 11

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[Problems] 8.1: 12(a,c), 23; 8.2: 4(b), 22; 8.3: 1(b).

8.1 - 12. For the initial value problem

$$\begin{aligned} y' &= (y^2 + 2ty)/(3 + t^2), \\ y(0) &= 0.5, \end{aligned}$$

find the approximate values of the solution at $t = 0.5, 1.0, 1.5$ and 2.0.

(a) Use the Euler method with $h = 0.025$.

(c) Use the backward Euler method with $h = 0.025$.

8.1 - 23. In this problem we discuss the global truncation error associated with the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. Assuming that the functions f and f_y are continuous in a closed, bounded region R of the ty-plane that includes the point (t_0, y_0), it can be shown that there exists a constant L such that $|f(t, y) - f(t, \tilde{y})| < L|y - \tilde{y}|$, where (t, y) and $(t, \tilde{y})$ are any two points in R with the same t coordinate. Further, we assume that f_t is continuous, so the solution ϕ has a continuous derivative.

(a) Using

$$\phi(t_{n+1}) = \phi(t_n) + \phi'(t_n)\, h + \frac{1}{2}\phi''(\bar{t}_n)\, h^2,$$

show that

$$\begin{aligned} |E_{n+1}| &\leqslant |E_n| + h\,|f[t_n, \phi(t_n)] - f(t_n, y_n)| + \frac{1}{2}h^2|\phi''(\bar{t}_n)| \qquad (1) \\ &\leqslant \alpha|E_n| + \beta h^2, \end{aligned}$$

where $\alpha = 1 + hL$ and $\beta = \max|\phi''(t)|/2$ on $t_0 \leqslant t \leqslant t_n$.

(b) Assume that if $E_0 = 0$ and if $|E_n|$ satisfies the Eqn (1), then $|E_n| \leqslant \beta h^2(\alpha^n - 1)/(\alpha - 1)$ for $\alpha \neq 1$. Use this result to show that

$$|E_n| \leqslant \frac{(1 + hL)^n - 1}{L}\beta h. \qquad (2)$$

Eqn (2) gives a bound for $|E_n|$ in terms of h, L, n and β. Notice that for a fixed h, this error bound increases with increasing n; that is, the error bound increases with distance from the starting point t_0.

(c) Show that $(1 + hL)^n \leqslant e^{nhL}$; hence

$$|E_n| \leqslant \frac{e^{nhL} - 1}{L}\beta h. \qquad (3)$$

If we select an ending point T greater than t_0 and then choose the step size h so that n steps are required to traverse the interval $[t_0, T]$, then $nh = T - t_0$, and

$$|E_n| \leqslant \frac{e^{(T-t_0)\,L} - 1}{L}\,\beta h = Kh. \tag{4}$$

Note that K depends on the length $T - t_0$ of the interval and on the constants L and β that are determined from the function f.

8.2 - 4. For the initial value problem

$$\begin{aligned} y' &= 2t + e^{-ty}, \\ y(0) &= 1, \end{aligned}$$

find the approximate values of the solution at $t = 0.1, 0.2, 0.3$ and 0.4. Compare the results with those obtained by the Euler method and with the exact solution (if available).

(b) Use the Euler method with $h = 0.025$.

8.2 - 22. The **modified Euler formula** for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ is given by

$$y_{n+1} = y_n + h f\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}h\, f(t_n, y_n)\right].$$

Show that the local truncation error in the modified Euler formula is proportional to h^3.

8.3 - 1. For the initial value problem

$$\begin{aligned} y' &= 2 + t - y, \\ y(0) &= 1, \end{aligned}$$

find the approximate values of the solution at $t = 0.1, 0.2, 0.3$ and 0.4. Compare the results with those obtained by the numerical method and with the exact solution (if available).

(b) Use the Runge-Kutta method with $h = 0.05$.