MATH 2352 Problem Sheet 10

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[Problems] 7.7: 6, 12, 13; 7.8: 7, 18, 22 7.9: 1, 3.

7.7 - 6. For

$$\boldsymbol{x}' = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}\boldsymbol{x},$$

(a) Find a fundamental matrix for the given system of equations.

(b) Also find the fundamental matrix $\boldsymbol{\Phi}(t)$ satisfying $\boldsymbol{\Phi}(0) = \boldsymbol{I}$.

7.7 - 12. Solve the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\boldsymbol{x},$$

$$\boldsymbol{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

by using the fundamental matrix $\boldsymbol{\Phi}(t)$ for the following system of equations:

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\boldsymbol{x}.$$

7.7 - 13. Show that $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\,\boldsymbol{\Psi}^{-1}(t_0)$, where $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Psi}(t)$ are as defined in this section (or as in the slides).

7.8 - 7. Consider the initial value problem

$$\boldsymbol{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\boldsymbol{x},$$

$$\boldsymbol{x}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

(a) Find the solution.

(b) Draw the trajectory of the solution in the $x_1 x_2$-plane, and also draw the graph of x_1 versus t.

7.8 - 18. Consider the system

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\boldsymbol{x}. \qquad (1)$$

(a) Show that $r=2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\boldsymbol{A}$ and that there is only one corresponding eigenvector, namely,

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Using the information in part (a), write down one solution $\boldsymbol{x}^{(1)}(t)$ of the system (1). There is no other solution of the purely exponential form $\boldsymbol{x} = \boldsymbol{\xi}\, e^{rt}$.

(c) To find a second solution, assume that $\boldsymbol{x} = \boldsymbol{\xi}\, t e^{2t} + \boldsymbol{\eta} e^{2t}$. Show that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfy the equations

$$\begin{aligned} (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{\xi} &= \boldsymbol{0}, \\ (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{\eta} &= \boldsymbol{\xi}. \end{aligned}$$

Since $\boldsymbol{\xi}$ has already been found in part (a), solve the second equation for $\boldsymbol{\eta}$. Neglect the multiple of $\boldsymbol{\xi}^{(1)}$ that appears in $\boldsymbol{\eta}$, since it leads only to a multiple of the first solution $\boldsymbol{x}^{(1)}$. Then write down a second solution $\boldsymbol{x}^{(2)}(t)$ of the system (1).

(d) To find a third solution, assumen that $\boldsymbol{x} = \boldsymbol{\xi}\,(t^2/2)\, e^{2t} + \boldsymbol{\eta}\, t e^{2t} + \boldsymbol{\zeta}\, e^{2t}$. Show that $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations

$$\begin{aligned} (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{\xi} &= \boldsymbol{0}, \\ (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{\eta} &= \boldsymbol{\xi}, \\ (\boldsymbol{A} - 2\boldsymbol{I})\,\boldsymbol{\zeta} &= \boldsymbol{\eta}. \end{aligned}$$

The first two equations are the same as in part (c), so solve the third equation for $\boldsymbol{\zeta}$, again neglecting the multiple of $\boldsymbol{\xi}^{(1)}$ that appears. Then write down a third solution $\boldsymbol{x}^{(3)}(t)$ of the system (1).

(e) Write down a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system (1).

(f) Form a matrix $\boldsymbol{T}$ with the eigenvector $\boldsymbol{\xi}^{(1)}$ in the first column and the generalized eigenvector $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ in the second and third solumns. Then find $\boldsymbol{T}^{-1}$ and form the product $\boldsymbol{J} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$. The matrix $\boldsymbol{J}$ is the Jordan form of A.

7.8 - 22. Let

$$\boldsymbol{J} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

where λ is an arbitrary real number.

(a) Find $\boldsymbol{J}^2$, $\boldsymbol{J}^3$, and $\boldsymbol{J}^4$.

(b) Use an inductive argument to show that

$$\boldsymbol{J}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & [n(n-1)/2]\,\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

(c) Determine $e^{\boldsymbol{J}t}$.

(d) Note that if you choose $\lambda = 2$, then the matrix $\boldsymbol{J}$ in this problem is the same as the matrix $\boldsymbol{J}$ in Problem 18(f), form the product $\boldsymbol{T}e^{\boldsymbol{J}t}$ with $\lambda = 2$. The resulting matrix is the same as the fundamental matrix $\boldsymbol{\Psi}(t)$ in Problem 18(e).

7.9 - 1. Find the general solution of the given system of equations.

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

7.9 - 3. Find the general solution of the given system of equations.

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}.$$