MATH 2352 Problem Sheet 09

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[Problems] 7.1: 5, 6, 8; 7.4: 4, 9; 7.5: 15, 17, 19; 7.6: 7, 15(a, b);

7.1 - 5. Transform the given eqaution into a system of first order equations.

$$u'' + 2u' + 4u = 2\cos 3t,$$

$$u(0) = 1, \qquad u'(0) = -2.$$

7.1 - 6. Transform the given eqaution into a system of first order equations.

$$u'' + p(t)\, u' + q(t)\, u = g(t),$$

$$u(0) = u_0, \qquad u'(0) = u_0'.$$

7.1 - 8. For

$$\begin{aligned} x_1' &= 3x_1 - 2x_2, \\ x_2' &= 2x_1 - 2x_2. \end{aligned}$$

$$\begin{aligned} x_1(0) &= 3, \\ x_2(0) &= 1. \end{aligned}$$

(a) Transform the given system into a single equaion of second order.

(b) Find x_1 and x_2 that also satisfy the given initial conditions.

(c) Sketch the graph of the solution in the x_1x_2-plane for $t \geqslant 0$.

7.4 - 4. If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y'' + p(t)y' + q(t)y = 0 \tag{1}$$

corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -p(t)x_2 - q(t)x_1. \end{aligned} \tag{2}$$

Show that if $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are fundamental set of solutions of Eqs.(2), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq.(1), then $W\big[y^{(1)}, y^{(2)}\big] = c\, W\big[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\big]$, where c is a nonzero constant.

Hint: $y^{(1)}(t)$ and $y^{(2)}(t)$ must be linear combinations of $x_{11}(t)$ and $x_{12}(t)$.

7.4 - 9. Let $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}$ be linearly independent solution of $\boldsymbol{x}' = \boldsymbol{P}(t)\,\boldsymbol{x}$, where $\boldsymbol{P}$ is continuous on $\alpha < t < \beta$.

(a) Show that any solution $\boldsymbol{x} = \boldsymbol{z}(t)$ can be written in the form

$$\boldsymbol{z}(t) = c_1 \boldsymbol{x}^{(1)}(t) + \ldots + c_n \boldsymbol{x}^{(n)}(t)$$

for suitable constants $c_1, \ldots, c_n$.

(b) Show that the expression for the solution $\boldsymbol{z}(t)$ in part (a) is unique.

7.5 - 15. For

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \boldsymbol{x}, \\ \boldsymbol{x}(0) &= (3, -1)^T. \end{aligned}$$

Solve the initial value problem. Describe the behavior of the solution as $t \to \infty$.

7.5 - 17. For

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \boldsymbol{x}, \\ \boldsymbol{x}(0) &= (2, 0, 3)^T. \end{aligned}$$

Solve the initial value problem. Describe the behavior of the solution as $t \to \infty$.

7.5 - 19. The system $t\boldsymbol{x}' = \boldsymbol{A}\,\boldsymbol{x}$ is analogous to the second order Euler equation. Assuming that $\boldsymbol{x} = \boldsymbol{\xi}\, t^r$, where $\boldsymbol{\xi}$ is a constant vector, show that $\boldsymbol{\xi}$ and r must satisfy $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\xi} = 0$ in order to obtain nontrivial solutions of the given differential equation.

7.6 - 7. For

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \boldsymbol{x}.$$

Express the general solution of the given system of equations in terms of real-valued functions.

7.6 - 15. For

$$\boldsymbol{x}' = \begin{pmatrix} 2 & \alpha \\ -5 & -2 \end{pmatrix} \boldsymbol{x},$$

the coefficient matrix contains a parameter α.

(a) Determine the eigenvalues in terms of α.

(b) Find the critical value or values of α where the qualitative nature of the phase portrait for the system changes.